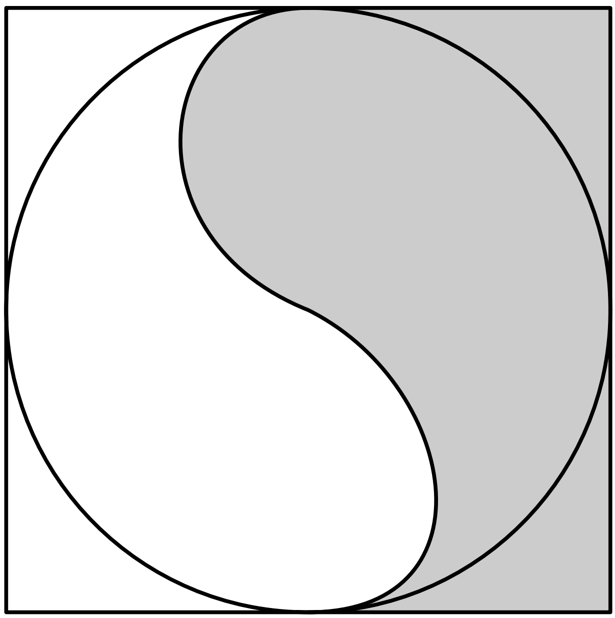
Differential Geometry of Spray and Finsler Spaces



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0.1 Introduction

In this book we study sprays and Finsler metrics. Roughly speaking, a spray on a manifold consists of compatible systems of second-order ordinary differential equations. A Finsler metric on a manifold is a family of norms in tangent spaces, which vary smoothly with the base point. Every Finsler metric determines a spray by its systems of geodesic equations. Thus, Finsler spaces can be viewed as special spray spaces. On the other hand, every Finsler metric defines a distance function by the length of minimial curves. Thus Finsler spaces can be viewed as regular metric spaces. Riemannian spaces are special regular metric spaces.

In 1854, B. Riemann introduced the Riemann curvature for Riemannian spaces in his ground-breaking *Habilitationsvortrag*. Thereafter the geometry of these special regular metric spaces is named after him. Riemann also mentioned general regular metric spaces, but he thought that there were nothing new in the general case. In fact, it is technically much more difficult to deal with general regular metric spaces. For more than half century, there had been no essential progress in this direction until P. Finsler did his pioneering work in 1918. Finsler studied the variational problems of curves and surfaces in general regular metric spaces. Some difficult problems were solved by him. Since then, such regular metric spaces are called Finsler spaces. Finsler, however, did not go any further to introduce curvatures for regular metric spaces. He switched his research direction to set theory shortly after his graduation.

It was L. Berwald who first successfully extended the notion of Riemann curvature to Finsler spaces. He also introduced a notion of non-Riemannian quantity—the Berwald curvature [Bw1]. From this point of view, Berwald is the founder of differential geometry of Finsler spaces. J. Douglas extended the Berwald curvature to general sprays. He also extended H. Weyl's projective invariant for affine connections to general sprays [Dg1].

Local geometric structures of sprays and Finsler metrics have been understood in a great depth after P. Finsler's pioneering work in 1918 (see [Fi] [Ca] [Ru] [Bu3] [Ma4], etc). The geometric methods developed in Finsler geometry are useful for studying some problems arising from biology, physics and other fields (see [AbPa] [AnZa1] [AnZa2] [AnInMa] [As1] [Bj] [Mi1][MiAn], etc). However, due to complicated tensor computations, spray geometry as well as Finsler geometry has made many beginners turn away from this subject. Indeed, one can easily get lost in the deep forest of tensors. In this book , we try to define quantities in an index-free form and derive equations using tensors only when necessary. We introduce several quantities of sprays or Finsler metrics such as the Riemann curvature, Berwald curvature and Landsberg curvature, etc. The names of these quantities are given for the first time, although these quantities have appeared in literatures.

In Chapter 1, we discuss some basic properties of Minkowski functionals and introduce the notion of Cartan torsion for Minkowski functionals. In particular, we show how a two-dimensional Minkowski functional depends on the main scalar (the core part of the Cartan torsion in dimension two). In Chapter 2, we define Finsler metrics and discuss many important examples. In Chapter 3, we show that the geodesics of a Finsler metric satisfy a system of second-order ordinary differential equations, that are the Euler-Lagrange equations of the variation of arc-length. We also discuss variational problems of first order and second-order ordinary differential equations. In Chapter 4, we introduce the notion of sprays, based on our discussion in the previous chapters. The spray induced by a Finsler metric is called a Finsler spray. By studying general sprays, we would have a better understanding of Finsler metrics.

In Chapters 5 and 6, we introduce several non-Riemannian quantities for sprays and Finsler metrics, which always vanish for Riemannian metrics. We calculate these quantities for some interesting sprays and Finsler metrics. We raise an open problem whether or not there are Landsberg metrics which are not of Berwald type. In Chapter 7, the concept of connection is introduced. Using the canonical connection, we define the parallelism along geodesics. In Chapter 8, we introduce the most important quantity — Riemann curvature for sprays and Finsler metrics. It is defined as the Jacobi endomorphism of the variation of geodesics. In Chapters 9 and 10, we derive several important equations for the quantities introduced in the previous chapters.

The flag curvature of Finsler metrics is the generalization of sectional curvature of Riemannian metrics. It is a function of a tangent plane and a vector (pole vector) in the plane. We say that a Finsler metric is of scalar curvature if the flag curvature is independent of the tangent planes containing the pole vector. When the flag curvature is constant, the Finsler metric is said to be of constant curvature. In Chapter 11, we prove several theorems for Finsler metrics of scalar curvature and constant curvature. Some interesting metrics of constant curvature are discussed as well.

In Chapter 12, we discuss Projective Geometry of spray and Finsler metrics. We prove the Rapcsák theorem which is one of the most important theorems in Projective Geometry. We derive an equation for the Riemann curvature under a projective change. In Chapter 13, we continue the discussion on projective geometry, and introduce two essential projective invariants, the Douglas curvature and the Weyl curvature. These two quantities play important roles in understanding the projective properties of sprays and Finsler metrics.

In Chapter 14, we study some global problems via the exponential map and Jacobi fields. We prove several comparison theorems for positive definite Finsler spaces and describe isotropic sprays and Finsler metrics of constant curvature.

The topics in this book are selected based on author's own interests. Some important topics and applications such as Lagrange geometry are only briefly discussed. The interested readers should consult books written by R. Miron and others [Mi1][MiAn].

Chapter 1

Minkowski Spaces

In this chapter, Minkowski spaces will be introduced. First, let us take a look at Euclidean spaces. Let R^n denote the n-dimensional canonical real vector space. Define

$$|y| := \sqrt{\sum_{i=1}^{n} |y^i|^2}, \qquad y = (y^i) \in \mathrm{R}^n. \tag{1.1}$$

$|\cdot|$ is called the *standard Euclidean norm* on R^n. We denote $\mathbb{R}^n = (\mathrm{R}^n, |\cdot|)$ which is called the *standard Euclidean space.* The *standard inner product* $\langle\cdot,\cdot\rangle$ on R^n is defined by

$$\langle u, v\rangle := \sum_{i=1}^{n} u^i v^i, \qquad u = (u^i),\ v = (v^i) \in \mathrm{R}^n.$$

This gives rise to the Euclidean norm

$$|y| = \sqrt{\langle y, y\rangle}.$$

More general, consider a nondegenerate bilinear symmetric form Q on an n-dimensional vector space V. Take a basis $\{e_i\}_{i=1}^n$ for V and express Q as follows

$$Q(u, v) := a_{ij} u^i v^j, \tag{1.2}$$

where $u = u^i e_i$, $v = v^i e_i$ and $\big(a_{ij}\big)$ is a symmetric matrix with $\det\big(a_{ij}\big) \neq 0$. This gives rise to a nondegenerate quadratic form on V

$$L(y) := Q(y, y). \tag{1.3}$$

Q is called an *inner product* in V if it is positive definite. In this case,

$$\alpha(y) = \sqrt{Q(y, y)}$$

is called an *Euclidean norm* on V and the pair (V, α) is called an *Euclidean space.* Every Euclidean space is linearly isometric to the standard Euclidean space $\mathbb{R}^n$.

1.1 Minkowski Functionals

We can generalize nondegenerate quadratic forms.

Definition 1.1.1 Let V be a finite dimensional vector space. A *Minkowski functional* on V is a function $L : \mathrm{V} \to \mathrm{R}$ which has the following properties:

(i) L is C^∞ on $\mathrm{V} \setminus \{0\}$,

(iia) L is positively homogeneous of degree two, i.e.,

$$L(\lambda y) = \lambda^2 L(y), \qquad \lambda > 0,\ y \in \mathrm{V} \tag{1.4}$$

(iib) for every $0 \neq y \in \mathrm{V}$, the *fundamental form* g_y on V is nondegenerate , where

$$g_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[L(y + su + tv)\Big]|_{s=t=0}. \tag{1.5}$$

A *Minkowski space* is a pair (V, L) where V is a finite dimensional vector space and L is a Minkowski functional on V.

By definition, if Q is a nondegenerate symmetric bilinear form on a vector space V, then

$$L(y) := Q(y, y)$$

is a Minkowski functional.

For a Minkowski functional L on a vector space V, the *indicatrix* of L

$$\mathrm{S} = \mathrm{S}_+ \cup \mathrm{S}_-$$

is defined by

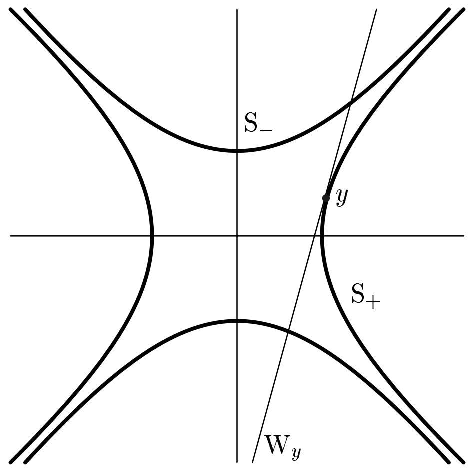
$$\mathrm{S}_\pm := \Big\{ y \in \mathrm{V},\ L(y) = \pm 1 \Big\}. \tag{1.6}$$

If L changes sign on V, then S is disconnected. In this case, the *cone* C of L is non-empty.

$$\mathrm{C} := \Big\{ y \in \mathrm{V},\ L(y) = 0 \Big\} \neq \emptyset. \tag{1.7}$$

For a vector $y \in \mathrm{S}$, the tangent space of S at y, $T_y\mathrm{S}$, can be identified with a hyperplane W_y in T_xM in a natural way, where

$$\mathrm{W}_y := \Big\{ w \in \mathrm{V},\ g_y(y, w) = 0 \Big\}. \tag{1.8}$$



Example 1.1.1 Define $L_\varepsilon : \mathrm{R}^2 \to \mathrm{R}$ by

$$L_\varepsilon(u, v) = u^2 + \varepsilon v^2,$$

where $\varepsilon \neq 0$. L_ε is a Minkowski functional on R^2. The indicatrix of L_{-1} is the hyperbola in R^2 and that of L_1 is the unit circle in R^2. ♯

Let V be an n-dimensional vector space and $\{e_i\}_{i=1}^n$ a basis for V. Then $L = L(y^i e_i)$ is a function of $(y^i) \in \mathrm{R}^n$. Put

$$g_{ij}(y) := \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j}(y). \tag{1.9}$$

Then

$$g_y(u, v) = g_{ij}(y) u^i v^j,$$

where $u = u^i e_i$ and $v = v^j e_j$. It follows from the homogeneity of L that

$$\frac{1}{2} \frac{\partial L}{\partial y^i}(y) = g_{ij}(y) y^j \tag{1.10}$$

$$L(y) = g_{ij}(y) y^i y^j. \tag{1.11}$$

Observe that the sign of the determinant $\det(g_{ij}(y))$ does not change on $\mathrm{V} \setminus \{0\}$. We can define

$$\mathrm{ind}\ (L) := \begin{cases} 1 & \text{if } \det(g_{ij}) > 0 \\ -1 & \text{if } \det(g_{ij}) < 0. \end{cases} \tag{1.12}$$

ind (L) is called the *index* of L.

Let $k(y)$ denote the number of negative eigenvalues of g_y. By continuity, we see that $k = k(y)$ is independent of $y \in \mathrm{V} \setminus \{0\}$ and the index of L is given by

$$\mathrm{ind}\,(L) = (-1)^k.$$

If $k = 0$, it follows from (1.11) that

$$L(y) > 0, \qquad y \in \mathrm{V} \setminus \{0\}.$$

In this case, L is said to be *positive definite* and (V, L) is called a *positive definite Minkowski space*. Usually, we denote a positive definite Minkowski functional by

$$F(y) := \sqrt{L(y)}.$$

The reader should consult A.C. Thompson [Tho] for comprehensive discussion on positive definite Minkowski functionals.

Example 1.1.2 Define $F : \mathrm{R}^n \to [0, \infty)$ by

$$F(y) := \sqrt{\sum_{i=1}^n \left(y^i\right)^2 + \lambda \sqrt{\sum_{i=1}^n \left(y^i\right)^4}},$$

where λ is an arbitrary nonnegative number. One can easily verify that F is a positive definite Minkowski functional on R^n. ♯

Example 1.1.3 There are many Minkowski functionals on a real vector space V in the following form

$$F(y) := \left[Q(y, y, y, y)\right]^{\frac{1}{4}}$$

where $Q = Q(u, v, w, z)$ is a symmetric multi-linear form on V. Consider the following functional on R^2

$$F(u, v) := \left\{u^4 + 2cu^2v^2 + v^4\right\}^{\frac{1}{4}}.$$

If $0 < c < 3$, then F is a positive definite Minkowski functional on R^2.

K. Okubo studied the following type of functionals

$$F(u, v) := \left\{\left(\lambda u^2 + \mu v^2\right)\left(\mu u^2 + \lambda v^2\right)\right\}^{\frac{1}{4}}, \tag{1.13}$$

where λ, μ are positive numbers. Rewriting it as follows

$$F(u, v) = (\lambda\mu)^{\frac{1}{4}} \left\{u^4 + 2\frac{\lambda^2 + \mu^2}{2\lambda\mu} u^2 v^2 + v^4\right\}^{\frac{1}{4}},$$

one can see that if λ and μ satisfy

$$3-2\sqrt{2}=\frac{1}{3+2\sqrt{2}}<\frac{\lambda}{\mu}<3+2\sqrt{2}=\frac{1}{3-2\sqrt{2}},$$

then F is a positive definite Minkowski functional.

One can also construct many Minkowski functionals on R^2 in the following form

$$F(u,v):=\sqrt{au^2+bv^2+\sqrt{u^4+2cu^2v^2+v^4}}+\lambda u$$

with suitable positive constants a, b and c. ♯

We have introduced *regular* Minkowski functionals on a vector space. The reality is that there are more singular Minkowski functionals than regular ones. Sometimes, we have to study singular Minkowski functionals in applications. Let $\mathcal{C}$ be an open cone in V, i.e., $\mathcal{C}\setminus\{0\}$ is an open subset satisfying

$$\lambda y\in\mathcal{C},\qquad \lambda>0,\ y\in\mathcal{C}.$$

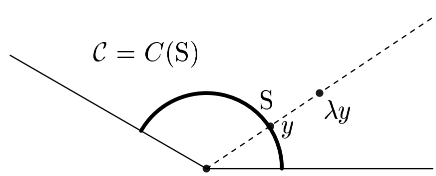
A Minkowski functional on V with regular conical domain $\mathcal{C}$ is a function $L:\mathcal{C}\to\mathrm{R}$ which has the following properties:

(i) L is C^∞ on $\mathcal{C}\setminus\{0\}$,

(iia) L is positively homogeneous of degree two, i.e.,

$$L(\lambda y)=\lambda^2L(y),\qquad \lambda>0,\quad y\in\mathcal{C},$$

(iib) for any $0\neq y\in\mathcal{C}$, the fundamental form g_y is nondegenerate.



Example 1.1.4 Let

$$F(y)=\Big\{\sum_{i=1}^n\left(y^i\right)^4\Big\}^{\frac{1}{4}},\qquad y\in\mathrm{R}^n.$$

F is a singular Minkowski functional on R^n. ♯

Example 1.1.5 (Berwald-Moòr functional) Define $L : \mathrm{R}^n \to \mathrm{R}$ by

$$L(y) = \Big(y^1 \cdots y^n\Big)^{\frac{2}{n}}. \tag{1.14}$$

A direct computation yields

$$g_{ij}(y) = \frac{L(y)}{n y^i y^j}\Big(\frac{2}{n} - \delta_{ij}\Big)$$

and

$$\det\Big(g_{ij}(y)\Big) = \frac{(-1)^n}{n^n}.$$

Thus L is a singular Minkowski functional. See [Ma4]. ♯

Example 1.1.6 Let $L(\alpha, \beta)$ be a positively homogeneous function of degree two on R^2, i.e.,

$$L(\lambda\alpha, \lambda\beta) = \lambda^2 L(\alpha, \beta), \qquad \lambda > 0, \ \alpha, \beta \in \mathrm{R}^2.$$

Let $\alpha(y) = \sqrt{a_{ij} y^i y^j}$ be an Euclidean norm and $\beta(y) = b_i y^i$ a one-form on a vector space V. Define

$$L(y) := L(\alpha(y), \beta(y))$$

$L(y)$ is called an *(α, β)-functional* on V. (α, β)-functionals have been studied by many Finslerian geometers. Recently, Sabau-Shimada [SaSh] made a classification of (α, β)-functionals. To determine whether L is a (singular) Minkowski functional, Matsumoto introduced several invariants [Ma4]

$$\rho = \frac{L_\alpha}{2\alpha}, \quad \rho_0 = \frac{L_{\beta\beta}}{2}, \quad \rho_{-1} = \frac{L_{\alpha\beta}}{2\alpha}, \quad \rho_{-2} = \frac{L_{\alpha\alpha}}{2\alpha^2} - \frac{L_\alpha}{2\alpha^3}.$$

Put

$$Y_i := a_{ij} y^j.$$

The fundamental form of L is given by

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_{-1}(b_i Y_j + b_j Y_i) + \rho_{-2} Y_i Y_j. \tag{1.15}$$

Let

$$\lambda := \frac{\rho_{-2}}{\rho} - \frac{\rho_0}{\rho}\Big(\frac{\rho_{-1}}{\rho_0}\Big)^2, \qquad \mu := \frac{\rho_{-1}}{\rho_0}, \qquad \delta := \frac{\rho_0}{\rho}.$$

We rewrite (1.15) as follows

$$g_{ij} = \rho\Big\{a_{ij} + \lambda Y_i Y_j + \delta(b_i + \mu Y_i)(b_j + \mu Y_j)\Big\}. \tag{1.16}$$

Consider a special (α, β)-functional

$$L = (\alpha + \beta)^2.$$

L is called a *Randers functional*. By an elementary argument, we obtain

$$\det(g_{ij}) = \Big(\frac{\alpha+\beta}{\alpha}\Big)^{n+1}\det(a_{ij}). \tag{1.17}$$

♯

Definition 1.1.2 Let V be an $(n-1)$-dimensional real vector space. A function $\phi = \phi(\xi)$ on V is called a *Lagrange functional* if it is C^∞ on V $- \{0\}$ and

$$\phi_{\xi^a\xi^b}(\xi) \neq 0, \qquad \xi \in \mathrm{V}.$$

In addition, if $\phi = \phi(\xi)$ is positively homogeneous of degree two in ξ, then ϕ is a Minkowski functional. In this sense, Lagrange functionals are generalized Minkowski functionals. However, Lagrange functionals are not essentially more general than Minkowski functionals.

Given an arbitrary functional ϕ on R^{n-1}, define

$$L(y) := \Big[y^1\phi\Big(\frac{y^2}{y^1},\cdots,\frac{y^n}{y^1}\Big)\Big]^2. \tag{1.18}$$

It is easy to see that ϕ is C^∞ on R^{n-1} if and only if L is C^∞ on $\mathrm{R}^n \setminus \{y^1 = 0\}$. The following lemma tells us that if ϕ is a Lagrange functional, then L is a singular Minkowski functional.

Lemma 1.1.3 *Let ϕ be a C^∞ function on R^{n-1} and L a C^∞ function on $R^n \setminus \{y^1 = 0\}$ given by (1.18). Let*

$$h_{ab}(\xi) := \frac{1}{2}\phi_{\xi^a\xi^b}(\xi) \qquad g_{ij}(y) = \frac{1}{2}L_{y^iy^j}(y).$$

Then at $y = (1, \xi)$ with $\phi(\xi) \neq 0$, the following are equivalent

$$\det\Big(\, h_{ab}(\xi)\,\Big) \neq 0 \quad \Longleftrightarrow \quad \det\Big(\, g_{ij}(y)\,\Big) \neq 0.$$

Proof: An easy computation yields that

$$\begin{pmatrix} g_{11} & g_{1b} \\ g_{a1} & g_{ab}\end{pmatrix} = \begin{pmatrix} \phi^2 - 2\phi\phi_d\xi^d + (\phi_d\xi^d)^2 + 2\phi h_{ab}\xi^a\xi^b & \phi\phi_b - \phi_b\phi_d\xi^d - 2\phi h_{bd}\xi^d \\ \phi\phi_a - \phi_a\phi_d\xi^d - 2\phi h_{ad}\xi^d & 2\phi h_{ab} + \phi_a\phi_b \end{pmatrix}$$

where $\phi_a := \frac{\partial\phi}{\partial\xi^a}(\xi)$.

Assume that $\det\Big(h_{ab}(\xi)\Big) = 0$. Then there is a vector $\zeta = \big(\zeta^2,\cdots,\zeta^n\big) \neq 0$ satisfying

$$h_{ab}(\xi)\zeta^b = 0. \tag{1.19}$$

Let

$$v = (v^1, \cdots, v^n) := \Big(\phi_a \zeta^a, \; -\phi \zeta^b + \phi_a \zeta^a \xi^b\Big).$$

Clearly, $v \neq 0$. One can easily verify that at $y = (1, \xi)$

$$g_{ij}(y) v^j = 0. \tag{1.20}$$

Thus $\det(g_{ij}(y)) = 0$.

Conversely, if $\det\Big(g_{ij}(y)\Big) = 0$ at $y = (1, \xi)$, then there is a vector $v = \big(v^1, \cdots, v^n\big) \neq 0$ satisfying (1.20). Let

$$\zeta = \big(\zeta^2, \cdots, \zeta^n\big) := \big(-v^2 + v^1 \xi^2, \cdots, -v^n + v^1 \xi^n\big).$$

We claim that

$$h_{ab}(\xi)\zeta^b = 0, \quad \text{and} \quad \zeta \neq 0. \tag{1.21}$$

To verify (1.21), we substitute $v^b = -\phi\zeta^b + \xi^b v^1$ for v^b in (1.20) and obtain

$$(\phi - \phi_d \xi^d) v^1 - \big[(\phi - \phi_d \xi^d)\phi_a \zeta^a - 2\phi h_{ab}\zeta^a \xi^b\big] = 0 \tag{1.22}$$

$$\phi_a v^1 - \big[2\phi h_{ab} + \phi_a \phi_b\big]\zeta^b = 0. \tag{1.23}$$

Contracting (1.23) with ξ^a yields

$$\phi_d \xi^d v^1 - \big[\phi_d \xi^d \phi_b \zeta^b + 2\phi h_{ab}\xi^a \zeta^b\big] = 0. \tag{1.24}$$

Adding (1.24) to (1.22) yields

$$v^1 - \phi_a \zeta^a = 0. \tag{1.25}$$

Plugging it into (1.23) gives $h_{ab}\zeta^b = 0$. From (1.25), we see that $\zeta \neq 0$, otherwise $v^1 = 0$, then $v^a = -\phi\zeta^a + \xi^a v^1 = 0$. This proves (1.21). Thus $\det(h_{ab}(\xi)) = 0$.
Q.E.D.

Example 1.1.7 Let $\phi : \mathrm{R}^{n-1} \to \mathrm{R}$ be given by

$$\phi(\xi) = A + A_a \xi^a + A_{ab}\xi^a \xi^b,$$

where $\det\Big(A_{ab}\Big) \neq 0$. Define

$$L(y) := \Big(\frac{A y^1 y^1 + A_a y^1 y^a + A_{ab} y^a y^b}{y^1}\Big)^2.$$

By Lemma 1.1.3, we know that for $y = (1, \xi) \in \mathrm{R}^n$ with $\phi(\xi) \neq 0$,

$$\det\Big(g_{ij}(y)\Big) \neq 0.$$

Thus L is a singular Minkowski functional. L is called a *Kropina functional* on R^n. A general Kropina functional on R^n is a singular Minkowski functional in the following form

$$L(y) = \Big(\frac{L_o(y)}{\beta(y)}\Big)^2$$

where L_o is a singular Minkowski functional and β is a 1-form on R^n.

Let $\bar{L}(\xi) = \bar{L}(\xi^2, \cdots, \xi^n)$ be a Minkowski functional on R^{n-1} and $A, A_2, \cdots, A_n$ be constants. Define

$$L(y) := \Big[Ay^1 + \sum_{a=2}^{n} A_a y^a + y^1 \bar{L}\Big(\frac{y^2}{y^1}, \cdots, \frac{y^n}{y^1}\Big)\Big]^2.$$

Then L is a singular Minkowski functional on R^n. ♯

1.2 Cartan Torsion

For a singular Minkowski functional L on a vector space V with regular conical domain $\mathcal{C}$, there is an important quantity introduced by P. Finsler [Fi] and emphased by E. Cartan [Ca]. Fix a basis $\{e_i\}_{i=1}^n$ for V. $L = L(y^i e_i)$ becomes a function on an open cone in R^n. For a vector $y \in \mathcal{C} \setminus \{0\}$, let

$$C_{ijk}(y) := \frac{1}{4} L_{y^i y^j y^k}(y). \tag{1.26}$$

Define $\mathbf{C}_y : V \otimes V \otimes V \to R$ by

$$\mathbf{C}_y(u, v, w) = C_{ijk}(y) u^i v^j w^k,$$

where $u = u^i e_i, v = v^j e_j$ and $w = w^k e_k$. $\mathbf{C}_y$ is a multi-linear symmetric form on V. It follows from the homogeneity of L that

$$\mathbf{C}_y(y, v, w) = 0. \tag{1.27}$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in \mathcal{C} \setminus \{0\}}$ is called the *Cartan torsion*. The following proposition is trivial, due to E. Cartan.

Proposition 1.2.1 ([Ca]) *For a Minkowski functional L on a vector space* V, $\mathbf{C} = 0$ *if and only if L is quadratic.*

For a vector $y \in \mathcal{C} \setminus \{0\}$, define $\mathbf{I}_y : V \to R$ by

$$\mathbf{I}_y(u) := \sum_{ij=1}^{n} \mathbf{C}_y(u, e_i, e_j) g^{ij}(y), \tag{1.28}$$

where $\{e_i\}_{i=1}^n$ be an arbitrary basis for V and $g_{ij}(y) := g_y(e_i, e_j)$. We call the family $\mathbf{I} := \{\mathbf{I}_y\}_{y\in\mathcal{C}\setminus\{0\}}$ the *mean Cartan torsion*. The mean Cartan torsion is an important quantity for Minkowski functionals.

Let $I_i(y) := \mathbf{I}_y(e_i)$. It is easy to verify the following

$$I_i(y) = g^{ij}(y)C_{ijk}(y) = \frac{\partial}{\partial y^i}\Big[\ln\sqrt{\det(g_{kl}(y))}\Big]. \tag{1.29}$$

Thus $\mathbf{I} = 0$ if and only if $\det(g_{ij}(y)) = constant$. There are many non-quadratic Minkowski functionals with $\mathbf{I} = 0$. For example, the Berwald-Moòr functional in (1.14) has vanishing mean Cartan torsion. It is proved by A. Deike that every positive definite Minkowski functional with $\mathbf{I} = 0$ must be Euclidean [De]. In [JiSh], we show that there are infinitely many non-Euclidean singular Minkowski functionals with $\mathbf{I} = 0$ which are defined on a small open cone in R^n.

For a positive definite Minkowski functional L on a vector space V, define

$$\|\mathbf{C}_y\| := \sup_{g_y(v,v)=1}\Big|\mathbf{C}_y(v,v,v)\Big|, \qquad \|\mathbf{C}\| := \sup_{L(y)=1}\|\mathbf{C}_y\|.$$

Note that $\mathbf{C} = 0$ if and only if $\|\mathbf{C}\| = 0$. Thus $\|\mathbf{C}\|$ measures the non-Riemannian features of L at certain degree.

Let $\alpha(y) = \sqrt{a_{ij}y^iy^j}$ be an Euclidean norm on a vector space V and $\beta(y) = b_iy^i$ a linear functional on V. Consider

$$F(y) := \alpha(y) + \beta(y).$$

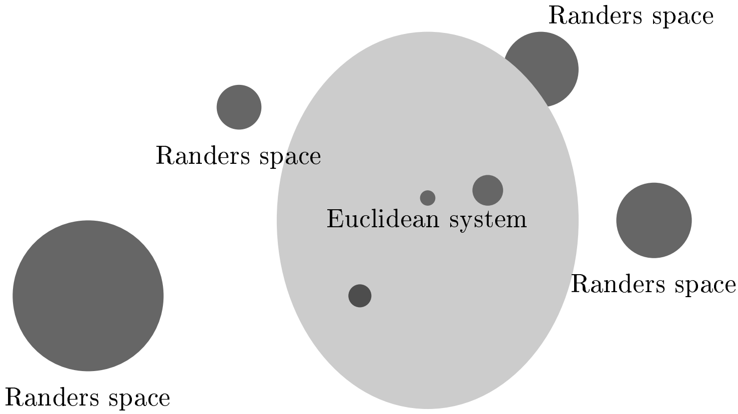
F or $L = F^2$ is called a *Randers functional*. Assume that

$$\|\beta\|_\alpha := \sup_{\alpha(y)=1}|\beta(y)| = \sqrt{a^{ij}b_ib_j} < 1,$$

where $\big(a^{ij}\big) := \big(a_{ij}\big)^{-1}$. Then it is easy to see that F is a positive definite Minkowski functional. By an elementary argument,

$$\|\mathbf{C}\| = \frac{3}{\sqrt{2}}\sqrt{1-\sqrt{1-\|\beta\|_\alpha^2}} \le \frac{3}{\sqrt{2}}. \tag{1.30}$$

See Example 1.2.2 or [BaChSh1] for a proof in dimension two. See also [Sh8] for a proof in higher dimensions. A Randers functional is a linear shift of an Euclidean norm. In the "universe" of Minkowski functionals ("planets"), Randers functionals are the "planets" closest to the "Euclidean system". No wonder, their Cartan torsions have uniform bound.

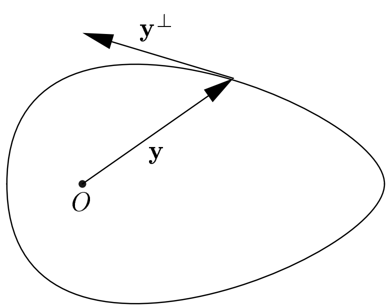


Minkowski Planes: For a (singular) Minkowski functional L on an oriented vector plane V, the Cartan torsion is determined by a scalar function on $\mathrm{V} \setminus \{0\}$. First we prove the following

Lemma 1.2.2 *Let* (V, L) *be a Minkowski plane. For a vector* $\mathbf{y} \in \mathrm{V}$ *with* $L(\mathbf{y}) \neq 0$, *there is a vector* $\mathbf{y}^{\perp} \in \mathrm{V} \setminus \{0\}$ *such that*

$$g_{\mathbf{y}}(\mathbf{y}, \mathbf{y}^{\perp}) = 0, \qquad g_{\mathbf{y}}(\mathbf{y}^{\perp}, \mathbf{y}^{\perp}) = \varepsilon\, L(\mathbf{y}), \tag{1.31}$$

where $\varepsilon = \mathrm{ind}\,(L)$ *denotes the index of* L *defined in (1.12). Moreover,* $\{\mathbf{y}, \mathbf{y}^{\perp}\}$ *is positively oriented if* $L(\mathbf{y}) > 0$ *and negatively oriented if* $L(\mathbf{y}) < 0$.



Proof of Lemma 1.2.2: Take an oriented basis $\{e_1, e_2\}$ for V. This basis determines a global coordinate system (u, v) in V. Let

$$L(u, v) := L(ue_1 + ve_2).$$

For a vector $\mathbf{y} = ue_1 + ve_2 \in \mathrm{V} \setminus \{0\}$, define $\mathbf{y}^\perp$ by

$$\mathbf{y}^\perp = \frac{-L_v\, e_1 + L_u\, e_2}{\sqrt{|L_{uu}L_{vv} - L_{uv}L_{uv}|}}. \tag{1.32}$$

Observe that

$$L_vL_{uu} - L_uL_{uv} = \Big[L_{uu}L_{vv} - L_{uv}L_{uv}\Big]v$$

$$-L_vL_{uv} + L_uL_{vv} = \Big[L_{uu}L_{vv} - L_{uv}L_{uv}\Big]u.$$

Thus

$$\begin{aligned} L_v^2L_{uu} - 2L_uL_vL_{uv} + L_u^2L_{vv} &= L_v[L_vL_{uu} - L_uL_{uv}] + L_u[-L_vL_{uv} + L_uL_{vv}] \\ &= (uL_u + vL_v)[L_{uu}L_{vv} - L_{uv}L_{uv}] \\ &= 2L[L_{uu}L_{vv} - L_{uv}L_{uv}]. \end{aligned}$$

Using these identities, one can easily verify that

$$\begin{aligned} g_{\mathbf{y}}(\mathbf{y}, \mathbf{y}^\perp) &= 0, \\ g_{\mathbf{y}}(\mathbf{y}^\perp, \mathbf{y}^\perp) &= \frac{L_v^2L_{uu} - 2L_uL_vL_{uv} + L_u^2L_{vv}}{2\Big|L_{uu}L_{vv} - L_{uv}L_{uv}\Big|} = \varepsilon L(y). \end{aligned}$$

Thus $\mathbf{y}^\perp$ satisfies (1.31). Q.E.D.

The pair $\{\mathbf{y}, \mathbf{y}^\perp\}$ is called the *Berwald frame* at $\mathbf{y}$. Define

$$\mathbf{I}(\mathbf{y}) := \frac{\mathbf{C}_{\mathbf{y}}(\mathbf{y}^\perp, \mathbf{y}^\perp, \mathbf{y}^\perp)}{L(y)} = \mathbf{I}_{\mathbf{y}}(\mathbf{y}^\perp). \tag{1.33}$$

From the definition, we see that $\mathbf{I}$ has the following homogeneity property:

$$\mathbf{I}(\lambda\mathbf{y}) = \mathbf{I}(\mathbf{y}), \qquad \lambda > 0,\ \mathbf{y} \in \mathrm{V} \setminus \{0\}.$$

We call $\mathbf{I}$ the *main scalar* of L.

Assume that L is a positive definite functional on a vector plane V. Hence

$$L_{uu}L_{vv} - L_{uv}L_{uv} > 0.$$

As in the proof of Lemma 1.2.2, we fix a basis $\{e_1, e_2\}$ for V and denote by (u, v) the corresponding global coordinate system. Then $L(u, v) = L(ue_1 + ve_2)$ is a function on the uv-plane. By the homogeneity of L, we have

$$L_{uu}L_{vv} - L_{uv}L_{uv} = \frac{1}{u^2}\Big(2LL_{vv} - L_vL_v\Big),$$

$$L_{uuu} = -\Big(\frac{v}{u}\Big)^3 L_{vvv}, \qquad L_{uuv} = \Big(\frac{v}{u}\Big)^2 L_{vvv}, \qquad L_{uvv} = -\Big(\frac{v}{u}\Big) L_{vvv}.$$

Observe that for $\mathbf{y} = ue_1 + ve_2$

$$\begin{aligned}
\mathbf{I}(\mathbf{y}) &= \frac{1}{4}\frac{L_{uuu}(-L_v)^3 + 3L_{uuv}(-L_v)^2 L_u + 3L_{uvv}(-L_v)(L_u)^2 + L_{vvv}(L_u)^3}{L\Big(L_{uu}L_{vv} - L_{uv}L_{uv}\Big)^{\frac{3}{2}}} \\
&= \frac{1}{4}\frac{u^3\Big(\frac{v}{u}L_v + L_u\Big)^3 L_{vvv}}{L\Big(2LL_{vv} - L_v L_v\Big)^{\frac{3}{2}}} \\
&= \frac{2L^2 L_{vvv}}{\Big(2LL_{vv} - L_v L_v\Big)^{\frac{3}{2}}}.
\end{aligned}$$

Thus

$$\mathbf{I}(\mathbf{y}) = \frac{2L^2 L_{vvv}}{\Big(2LL_{vv} - L_v L_v\Big)^{\frac{3}{2}}}. \tag{1.34}$$

Express L as

$$L(u,v) = \Big[u\phi\Big(\frac{v}{u}\Big)\Big]^2,$$

where $\phi = \phi(\xi)$ is a positive C^∞ function with $\phi_{\xi\xi}(\xi) > 0$. Then for $\mathbf{y} = e_1 + \xi e_2$

$$\mathbf{I}(\mathbf{y}) = \frac{3\phi_\xi \phi_{\xi\xi} + \phi\phi_{\xi\xi\xi}}{2\phi^{\frac{1}{2}}(\phi_{\xi\xi})^{\frac{3}{2}}}. \tag{1.35}$$

Example 1.2.1 Let

$$L(u,v) := \frac{v^{2+2a}}{u^{2a}} = \Big[u\Big(\frac{v}{u}\Big)^{1+a}\Big]^2, \qquad u > 0,\ v > 0,$$

where $a > 0$. Plugging $\phi = \xi^{1+a}$ into (1.35) gives

$$\mathbf{I} = \frac{1+2a}{\sqrt{a(1+a)}} = constant.$$

Note that

$$\min_{a>0} \mathbf{I} = 2, \qquad \max_{a>0} \mathbf{I} = \infty.$$

♯

Example 1.2.2 Let

$$F(u,v) = \sqrt{u^2+v^2} + Bu = |u|\Big\{\sqrt{1+\Big(\frac{v}{u}\Big)^2} + \varepsilon B\Big\},$$

where $|B| < 1$ and $\varepsilon = \pm$ (the sign of u). Plugging $\phi = \sqrt{1+\xi^2} + \varepsilon B$ into (1.35), we obtain the main scalar at $\mathbf{y} = (1, \xi)$

$$\mathbf{I} = -\frac{3}{2} \frac{\varepsilon B \xi}{\left(1+\xi^2\right)^{1/4} \sqrt{\sqrt{1+\xi^2} + \varepsilon B}}.$$

It is easy to see that

$$\max |\mathbf{I}| = \frac{3}{\sqrt{2}} \sqrt{1 - \sqrt{1 - B^2}}.$$

Clearly, the above estimate holds for any two-dimensional Randers functional $F = \alpha + \beta$ with $B := \|\beta\|_\alpha < 1$. Note that in dimension two,

$$\|\mathbf{C}\| = \max |\mathbf{I}|.$$

Thus

$$\|\mathbf{C}\| \leq \frac{3}{\sqrt{2}} \sqrt{1 - \sqrt{1 - B^2}}. \tag{1.36}$$

The bound (1.36) is actually true for Randers functionals in any dimension. One can prove this using dimension reduction method. The proof is left to the reader. ♯

Let $L = F^2$ be a positive definite Minkowski functional on $\mathrm{V} = \mathrm{span}\{e_1, e_2\}$. Parameterize $C := F^{-1}(1)$ by $c(s)$ such that

$$g_{c(s)}(\dot{c}(s), \dot{c}(s)) = 1.$$

Let

$$\mathbf{I}(s) := \mathbf{I}(c(s)).$$

According to O. Varga [Va], the vector-valued function $c(s)$ satisfies the following equation

$$c''(s) + \mathbf{I}(s) c'(s) + c(s) = 0. \tag{1.37}$$

See Section 1.3 below for a proof.

Below we are going to express Minkowski functionals for a prescribed main scalar $\mathbf{I}(s)$. Let $\phi(s)$ and $\psi(s)$ be two linearly independent solutions of the following equation

$$y''(s) + \mathbf{I}(s) y'(s) + y(s) = 0. \tag{1.38}$$

Put

$$c(s) := \phi(s) e_1 + \psi(s) e_2.$$

Then $c(s)$ satisfies (1.37). Assume that $\psi(s) \neq 0$ and $\chi(s) := \phi(s)/\psi(s)$ has the inverse on an interval. Define

$$F\Big(u e_1 + v e_2\Big) := \frac{u}{\phi \circ \chi^{-1}\left(\frac{u}{v}\right)}. \tag{1.39}$$

Let I denote the range of χ and

$$\mathcal{C} := \Big\{ ue_1 + ve_2 \in \mathrm{V},\ \frac{u}{v} \in I,\ F(ue_1 + ve_2) > 0 \Big\}.$$

F is a positive function on $\mathcal{C}$.

Lemma 1.2.3 *The function F defined in (1.39) is a positive definite Minkowski functional on $\mathcal{C}$ and the main scalar of F is $\mathbf{I}(s)$.*

Proof: Clearly, F is positively homogeneous of degree one. Let $ue_1 + ve_2 \in \mathcal{C}$ satisfy $F\Big(ue_1 + ve_2\Big) = 1$ and $s = \chi^{-1}(\frac{u}{v})$. Then

$$u = \phi \circ \chi^{-1}(\frac{u}{v}) = \phi(s).$$

Since

$$\chi(s) = \frac{\phi(s)}{\psi(s)} = \frac{u}{v},$$

we obtain

$$v = \psi(s).$$

This implies $ue_1 + ve_2 = c(s)$. Therefore $F^{-1}(1)$ is the curve c. One can prove that F is a positive definite Minkowski functional on $\mathcal{C}$ and the main scalar of F at $c(s)$ is just $\mathbf{I}(s)$. The proof is left to the reader. Q.E.D.

Now we give the list of all singular positive definite Minkowski planes with constant main scalar. This is due to [Bw5]. We divide the problem into three cases.

Case 1: $\mathbf{I}^2 > 4$. The general solution of (1.37) is given by

$$c(s) = e^{-\frac{1}{2}(\mathbf{I}+\sqrt{\mathbf{I}^2-4})s}\mathbf{a} + e^{-\frac{1}{2}(\mathbf{I}-\sqrt{\mathbf{I}^2-4})s}\mathbf{b}, \tag{1.40}$$

where $\mathbf{a}$ and $\mathbf{b}$ are linear independent constant vectors. The Minkowski functional F is given by

$$F(y) = \beta_1(y)^{\frac{1}{2}+\frac{\mathbf{I}}{2\sqrt{\mathbf{I}^2-4}}}\beta_2(y)^{\frac{1}{2}-\frac{\mathbf{I}}{2\sqrt{\mathbf{I}^2-4}}}, \tag{1.41}$$

where $\beta_1, \beta_2 \in \mathrm{V}^*$.

Case 2: $\mathbf{I}^2 = 4$. The general solution of (1.37) is given by

$$c(s) = e^{-\frac{1}{2}\mathbf{I}s}\mathbf{a} + se^{-\frac{1}{2}\mathbf{I}s}\mathbf{b}, \tag{1.42}$$

where $\mathbf{a}$ and $\mathbf{b}$ are linear independent constant vectors. The Minkowski functional F is given by

$$F(y) = \beta_2(y)\exp\Big\{\frac{\mathbf{I}}{2}\frac{\beta_1(y)}{\beta_2(y)}\Big\}. \tag{1.43}$$

Case 3: $\mathbf{I}^2 < 4$. The general solution of (1.37) is given by

$$c(s) = e^{-\frac{1}{2}\mathbf{I}s} \cos\Big(\frac{1}{2}\sqrt{4-\mathbf{I}^2}\ s\Big)\mathbf{a} + e^{-\frac{1}{2}\mathbf{I}s} \sin\Big(\frac{1}{2}\sqrt{4-\mathbf{I}^2}\ s\Big)\mathbf{b}, \tag{1.44}$$

where $\mathbf{a}$ and $\mathbf{b}$ are linear independent constant vectors. The Minkowski functional F is given by

$$F(y) = \sqrt{\beta_1(y)^2 + \beta_2(y)^2}\exp\Big\{\frac{\mathbf{I}}{\sqrt{4-\mathbf{I}^2}}\tan^{-1}\Big(\frac{\beta_1(y)}{\beta_2(y)}\Big)\Big\}. \tag{1.45}$$

The formulas of Minkowski functionals with constant main scalar are also given in [MaSh1][MaSh2].

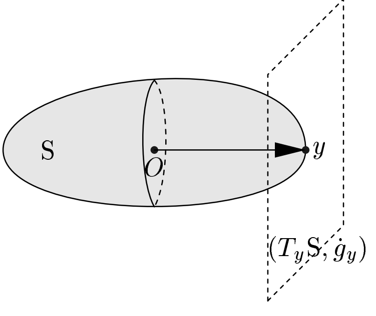
1.3 Varga Equation

In the previous section, we mentioned the Varga equation (1.37) for two-dimensional Minkowski functionals. In this section, we will prove the Varga equation for Minkowski functionals in all dimensions. Readers who are not familiar with geodesics and the Levi-Civita connection in Riemannian geometry can skip this section.

Every Minkowski functional L on a vector space V induces a Riemannian metric $\hat{g}$ on $\mathrm{V}\setminus\{0\}$ as follows. Define

$$\hat{g}_y(u,v) := g_y(u,v), \qquad u,v \in T_y\mathrm{V} \approx \mathrm{V}. \tag{1.46}$$

The indicatrix $\mathrm{S} = L^{-1}(1) \cup L^{-1}(-1)$ of L is a hypersurface in V.



For a point $y \in \mathrm{S}$, we can identify $T_y\mathrm{S}$ with the following hyperplane W_y in V

$$\mathrm{W}_y := \Big\{v \in \mathrm{V},\ g_y(y,v) = 0.\Big\}.$$

Define $\dot{g}_y : T_y\mathrm{S} \times T_y\mathrm{S} \to \mathrm{R}$ by

$$\dot{g}_y(u,v) = g_y(u,v), \qquad u,v \in T_y\mathrm{S} \approx \mathrm{W}_y. \tag{1.47}$$

$\dot{g} := \{\dot{g}_y\}_{y\in \mathrm{S}}$ is a Riemannian metric on S. Define $\mathbf{C}_y : T_y\mathrm{S} \otimes T_y\mathrm{S} \to T_y\mathrm{S}$ by

$$g_y\Big(\mathbf{C}_y(u,v), w\Big) := \mathbf{C}_y(u,v,w), \qquad u,v,w \in T_y\mathrm{S} \approx \mathrm{W}_y. \tag{1.48}$$

$\mathbf{C} = \{\mathbf{C}_y\}_{y\in \mathrm{S}}$ is called the *Cartan tensor* on S.

Let $\varphi : \mathrm{S} \to \mathrm{V}$ denote the natural embedding. We view φ as a vector-valued function on S. Let (u^a) be a local coordinate system in S. At $y = \varphi(u) \in \mathrm{S}$, $\dot{g}_{ab} := \dot{g}_y(\frac{\partial}{\partial u^a}, \frac{\partial}{\partial v^b})$ are given by

$$\dot{g}_{ab} := g_{ij}(\varphi(u))\varphi^i_a\varphi^j_b \tag{1.49}$$

and $C^c_{ab}\frac{\partial}{\partial u^c} := \mathbf{C}_y(\frac{\partial}{\partial u^a}, \frac{\partial}{\partial v^b})$ are given by

$$C^c_{ab} := \dot{g}^{cd}C_{ijk}(\varphi(u))\varphi^i_a\varphi^j_b\varphi^k_d. \tag{1.50}$$

where $\varphi^i_a := \frac{\partial \varphi^i}{\partial u^a}$.

For a smooth function f on S, define the *Hessian* $\dot{\mathrm{D}}^2 f$ by

$$\dot{\mathrm{D}}^2 f(u,v) := f_{a;b}u^a v^b, \qquad u = u^a\frac{\partial}{\partial u^a}, \;\; v = v^b\frac{\partial}{\partial u^b},$$

where $f_{a;b}$ are given by

$$f_{a;b} := \frac{\partial^2 f}{\partial u^a \partial u^b} - \gamma^c_{ab}\frac{\partial f}{\partial u^c},$$

where γ^c_{ab} denote the Christoffel symbols of $\dot{g}$.

Proposition 1.3.1 ([Va]) *The natural embedding* $\varphi = \varphi_\pm : \mathrm{S}_\pm = L^{-1}(\pm) \to \mathrm{V}$ *satisfies the following equation*

$$\dot{\mathrm{D}}^2\varphi + d\varphi(\mathbf{C}) \pm \dot{g}\,\varphi = 0. \tag{1.51}$$

Proof: We begin with following identity

$$g_{ij}(\varphi)\varphi^i\varphi^j = L(\varphi) = \pm 1. \tag{1.52}$$

Let $\varphi_a := \frac{\partial \varphi}{\partial u^a}$ and $\varphi_{ab} := \frac{\partial^2\varphi}{\partial u^a \partial u^b}$. Differentiating (1.52) yields

$$g_{ij}(\varphi)\varphi^i\varphi^j_a = 0, \tag{1.53}$$
$$g_{ij}(\varphi)\varphi^i_a\varphi^j_b + g_{ij}(\varphi)\varphi^i\varphi^j_{ab} = 0. \tag{1.54}$$

Express

$$\varphi^i_{ab} = \Lambda^c_{ab}\varphi^i_c + \lambda_{ab}\varphi^i. \tag{1.55}$$

It follows from (1.49), (1.53) and (1.54) that

$$\lambda_{ab} \pm \dot{g}_{ab} = 0. \tag{1.56}$$

Differentiating (1.49) yields

$$\frac{\partial \dot{g}_{ab}}{\partial u^c} = 2C_{abc} + \dot{g}_{ad}\Lambda^d_{bc} + \dot{g}_{bd}\Lambda^d_{ac}. \tag{1.57}$$

Since $\Lambda^a_{bc} = \Lambda^a_{cb}$, from (1.57), we obtain

$$\Lambda^a_{bc} = \gamma^a_{bc} - C^a_{bc}. \tag{1.58}$$

Plugging (1.56) and (1.58) into (1.55) gives (1.51). Q.E.D.

The Varga equation plays an important role in the study of Minkowski functionals. The Varga equation can be generalized to hypersurfaces in a Minkowski space. See [Sh4] .

Chapter 2

Finsler Spaces

The study of Finsler metrics began with P. Finsler's dissertation [Fi] in 1918. Finsler's thesis was was published in 1951. In his thesis, Finsler mainly studied the variational problems of Finsler metrics. He did not introduce the notions of curvatures which are the central concepts in geometry.

In this chapter, we will first discuss some basic facts on Finsler spaces — manifolds equipped with a Finsler metric.

2.1 Finsler Metrics

Let M be an n-dimensional manifold. The tangent bundle TM consists of all tangent vectors on M with the natural manifold structure. Let $\pi : TM \to M$ denote the natural projection. Let $(\mathcal{U}, \varphi)$ be a local coordinate system in M. Namely, $\mathcal{U}$ is an open subset of M and

$$\varphi = (\varphi^1, \cdots, \varphi^n) : \mathcal{U} \to \mathrm{R}^n$$

is a diffeomorphism onto an open subset of R^n. φ^i's are functions on $\mathcal{U}$ and their values $x^i = \varphi^i(x)$ at a point $x \in \mathcal{U}$ are called the coordinates of x. Such a map φ is called a coordinate map on M. The coordinate map φ induces a map

$$\hat{\varphi} = (\hat{\varphi}^1, \cdots, \hat{\varphi}^{2n}) : \hat{\mathcal{U}} := \pi^{-1}(\mathcal{U}) \to \mathrm{R}^n \times \mathrm{R}^n$$

which is defined by

$$\hat{\varphi}(y) = \Big(x^1, \cdots, x^n, y^1, \cdots, y^n\Big),$$

where $y = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$ and $\varphi(x) = (x^1, \cdots, x^n)$. $\hat{\varphi}$ is a diffeomorphism from $\hat{\mathcal{U}}$ onto an open subset in R^{2n}. We call $(\hat{\mathcal{U}}, \hat{\varphi})$ the *standard local coordinate system* in TM. For simplicity, we usually let (x^i) stand for $(\mathcal{U}, \varphi)$ and (x^i, y^i) for $(\hat{\mathcal{U}}, \hat{\varphi})$. Sometimes, we do not distinguish a point $x \in M$ with its coordinates (x^i) and a vector $y \in TM$ with its coordinates (x^i, y^i), if no confusion is caused.

The coordinate map φ induces n linear independent vector fields on $\mathcal{U}$ denoted by $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$, and the standard local coordinate map $\hat{\varphi}$ induces $2n$ linear independent vector fields on $\hat{\mathcal{U}}$ denoted by $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}_{i=1}^n$. For a scalar function L on TM, $L(y)$ is viewed as a function of (x^i, y^i) in a standard local coordinate system, not a function of (y^i), otherwise indicated.

Now we define Finsler metrics on a manifold.

Definition 2.1.1 Let M be a manifold. A *Finsler metric* on M is a function $L : TM \to \mathrm{R}$ which has the following properties

(i) L is C^∞ on $TM \setminus \{0\}$,

(ii) for any $x \in M$, the restriction $L_x := L|_{T_xM}$ is a Minkowski functional on T_xM, namely, the following two conditions are satisfied

(iia) L_x is positively homogeneous of degree two, i.e.,

$$L_x(\lambda y) = \lambda^2 L_x(y), \qquad \lambda > 0,\ y \in T_xM \setminus \{0\} \tag{2.1}$$

(iib) for any $y \in T_xM \setminus \{0\}$, the fundamental form g_y on T_xM is nondegenerate, where

$$g_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[L_x(y + su + tv)\Big]|_{s=t=0}, \qquad u, v \in T_xM.$$

A *Finsler space* is a pair (M, L), where M is a manifold and L is a Finsler metric on M.

Let (M, L) be a Finsler space. Take a basis $\{e_i\}_{i=1}^n$ for T_xM. $L(y) = L(y^i e_i)$ is a function on R^n. Let

$$g_{ij}(y) := \frac{1}{2} L_{y^i y^j}(y), \qquad y \in T_xM \setminus \{0\}. \tag{2.2}$$

Then

$$g_y(u, v) = g_{ij}(y) u^i v^j,$$

where $u = u^i e_i$ and $v = v^i e_i$. It follows from the homogeneity of L that

$$\frac{1}{2} L_{y^i}(y) = g_{ij}(y) y^j \tag{2.3}$$

$$L(y) = g_{ij}(y) y^i y^j. \tag{2.4}$$

If we take a natural basis $e_i := \frac{\partial}{\partial x^i}$ in a coordinate neighborhood $\mathcal{U}$, then $g_{ij}(y)$ are C^∞ functions on $\pi^{-1}(\mathcal{U}) \setminus \{0\} \approx \mathcal{U} \times (\mathrm{R}^n \setminus \{0\})$. Thus $\det(g_{ij}(y))$

does not change the sign on $\pi^{-1}(\mathcal{U})\setminus\{0\}$. Further, $\det(g_{ij}(y))$ does not change the sign when the coordinate system is changed. Thus

$$\mathrm{ind}\ (L) := \begin{cases} 1 & \text{if } \det(g_{ij}) > 0, \\ -1 & \text{if } \det(g_{ij}) < 0. \end{cases} \tag{2.5}$$

is well-defined. We call it the *index* of L. Let $k(y)$ denote the number of negative eigenvalues of $\Big(g_{ij}(y)\Big)$. By continuity, $k = k(y)$ is independent of $y \in TM\setminus\{0\}$. Moreover,

$$\mathrm{ind}\ (L) = (-1)^k.$$

When $k = 0$, it follows from (2.4) that

$$L(y) > 0, \qquad y \in TM \setminus \{0\}.$$

In this case, we call L a *positive definite Finsler metric* and (M, L) a *positive definite Finsler space*. We usually denote a positive definite Finsler metric L by its square root

$$F(y) := \sqrt{L(y)}.$$

Let (M, L) be a Finsler space. Since every $L_x := L|_{T_xM}$ is a Minkowski functional on T_xM. Fix an arbitrary basis $\{e_i\}_{i=1}^n$ for T_xM. Then $L = L(y^ie_i)$ is a function of $(y^i) \in \mathrm{R}^n$. For a vector $y = y^ie_i \in T_xM$, let

$$C_{ijk}(y) := \frac{1}{4}L_{y^iy^jy^k}(y). \tag{2.6}$$

Define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathrm{R}$ by

$$\mathbf{C}_y(u, v, w) := C_{ijk}(y)u^iv^jw^k,$$

where $u = u^ie_i$, $v = v^ie_i$ and $w = w^ie_i$. It follows from the homogeneity of L that

$$\mathbf{C}_y(y, v, w) = 0. \tag{2.7}$$

We call the family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM\setminus\{0\}}$ the *Cartan torsion* of L. Similarly, we define the *mean Cartan torsion* $\mathbf{I}_y : T_xM \to \mathrm{R}$ by

$$\mathbf{I}_y(u) = I_i(y)u^i, \qquad u = u^ie_i,$$

where

$$I_i(y) := g^{jk}(y)C_{ijk}(y).$$

It follows from (2.7) that

$$\mathbf{I}_y(y) = 0.$$

For oriented Finsler surfaces, we can also define the essential part of the (mean) Cartan torsion — the *main scalar* $\mathbf{I}$ as in (1.33) .

Our first important class of Finsler metrics are Riemannian metrics ($\mathbf{C} = 0$).

Example 2.1.1 Let $g = \{g_x\}_{x\in M}$, where g_x is a nondegenerate symmetric bilinear form in T_xM such that in local coordinates (x^i),

$$g_{ij}(x) = g_x\Big(\frac{\partial}{\partial x^i}|_x, \frac{\partial}{\partial x^j}|_x\Big)$$

are C^∞ functions. Let k denote the number of negative eigenvalues of g_x. g is called a *pseudo-Riemannian metric* if $k > 0$ and a *Riemannian metric* if $k = 0$. Put

$$L(y) := g(y, y), \qquad y \in TM.$$

L is a Finsler metric. When g is positive definite, we usually denote it by

$$\alpha(y) := \sqrt{g(y, y)}, \qquad y \in TM.$$

♯

Let $|\cdot|$ and $\langle , \rangle$ be the standard Euclidean norm and inner product in R^n. We have the following interesting Riemannian metrics defined on R^n or B^n.

$$L = \frac{|y|^2 - \Big(|x|^2|y|^2 - \langle x, y\rangle^2\Big)}{1 - |x|^2}, \tag{2.8}$$

$$L = \frac{|y|^2 + \Big(|x|^2|y|^2 - \langle x, y\rangle^2\Big)}{\Big(1 + |x|^2\Big)^2}, \tag{2.9}$$

$$L = \frac{|y|^2 + \Big(|x|^2|y|^2 - \langle x, y\rangle^2\Big)}{1 + |x|^2}, \tag{2.10}$$

$$L = \frac{|y|^2 - \Big(|x|^2|y|^2 - \langle x, y\rangle^2\Big)}{\Big(1 - |x|^2\Big)^2}, \tag{2.11}$$

$$L = \frac{4|y|^2}{\Big(1 - |x|^2\Big)^2}. \tag{2.12}$$

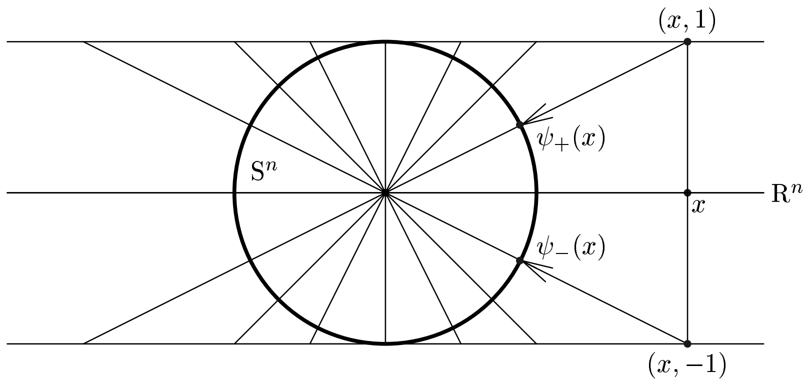
All these Riemannian metrics have special properties. Riemannian metrics in (2.8) and (2.9) have constant curvature = 1 and these in (2.10), (2.11) and (2.12) have constant curvature = −1. The notion of curvature will be defined later. The metric in (2.11) is called the *Klein metric* and the metric in (2.12) is called the *Poincare metric*.

Let S^n_+ and S^n_- denote the standard unit upper semisphere and lower semisphere in R^{n+1} respectively. Define a map

$$\psi_\pm : \mathrm{R}^n \to \mathrm{S}^n_\pm \subset \mathrm{R}^{n+1}$$

by

$$\psi_\pm(x) := \Big(\frac{x}{\sqrt{1 + |x|^2}}, \frac{\pm 1}{\sqrt{1 + |x|^2}}\Big). \tag{2.13}$$



ψ_+ and ψ_- cover S^n_+ and S^n_- respectively. The pull-back of the standard Riemannian metric on $\mathbb{S}^n$ by $\psi_\pm$ is just the metric in (2.9). We can deform the metric in (2.9) to get non-trivial Finsler metrics on R^n and S^n. See Examples 2.2.1 and 2.2.2 below.

There are several important Finsler metrics arising in physics, among them are Randers metrics (see [AnInMa]).

Example 2.1.2 (Randers metrics) Let $\alpha = \alpha(y)$ be a Riemannian metric on a manifold M and $\beta = \beta(y)$ be an 1-form on M. Suppose that

$$\|\beta\|_\alpha := \sup_{\alpha(v)=1} \beta(v) < 1.$$

Define

$$F(y) := \alpha(y) + \beta(y). \tag{2.14}$$

F is a positive definite Finsler metric. Randers metrics were introduced in the unified field theory and have been the basis of various branches of theoretical physics [AnInMa].

The following Randers metrics are very special. All of them are isometric to each other. They are the deformation of special Riemannian metrics in (2.10)-(2.12).

$$F_\varepsilon : \ = \ \sqrt{\frac{|y|^2 + \Big(|x|^2|y|^2 - \langle x, y\rangle^2\Big)}{4(1+|x|^2)}} \pm \varepsilon \frac{\langle x, y\rangle}{2(1+|x|^2)}, \tag{2.15}$$

$$F_\varepsilon : \ = \ \frac{\sqrt{|y|^2 - \Big(|x|^2|y|^2 - \langle x, y\rangle^2\Big)}}{2(1-|x|^2)} \pm \varepsilon \frac{\langle x, y\rangle}{2(1-|x|^2)}, \tag{2.16}$$

$$F_\varepsilon : \ = \ \frac{|y|}{1-|x|^2} \pm \varepsilon \frac{2\langle x, y\rangle}{1-|x|^4}, \tag{2.17}$$

where $|\varepsilon| \leq 1$. F_ε is of constant curvature -1 when $\varepsilon = 1$. The pair of metrics in (2.16) with $\varepsilon = 1$ are called the *Funk metrics* on the unit ball. It is proved by T. Okada [Ok] that the Funk metrics have constant curvature -1 (compare [Fu1] [Bw3]). More details will be given in Sections 2.3 below. ♯

Example 2.1.3 Let $c = c(x, y)$ and $\rho = \rho(x, y)$ be smooth functions on open domain $\Omega \subset \mathrm{R}^2$. Assume that c satisfies

$$0 < c(x, y) < 3.$$

Then, according to Example 1.1.3, the following function on $T\Omega = \Omega \times \mathrm{R}^2$

$$F := e^{\rho(x,y)} \Big\{ u^4 + 2c(x, y)u^2v^2 + v^4 \Big\}^{\frac{1}{4}} \tag{2.18}$$

is a positive definite Finsler metric. The author learned from Matsumoto that, In 1977, K. Okubo studied the following type of Finsler metrics on an open domain Ω in R^2

$$F = \Big\{ \Big(\lambda(x, y)u^2 + \mu(x, y)v^2\Big) \Big(\mu(x, y)u^2 + \lambda(x, y)v^2\Big) \Big\}^{\frac{1}{4}}, \tag{2.19}$$

where $\lambda(x, y)$ and $\mu(x, y)$ are positive functions. As shown in Example 1.1.3, Okubo's metrics can be expressed in the form (2.18). Thus, they are positive definite Finsler metrics if $\lambda(x, y)$ and $\mu(x, y)$ satisfy

$$3 - 2\sqrt{2} = \frac{1}{3 + 2\sqrt{2}} < \frac{\lambda(x, y)}{\mu(x, y)} < \frac{1}{3 - 2\sqrt{2}} = 3 + 2\sqrt{2}.$$

♯

We also come across lots of singular Finsler metrics in applications. Let $L : TM \to \mathrm{R}$ be a function satisfying (2.1). Let $\mathcal{C}M \subset TM \setminus \{0\}$ be an open cone, i.e., $\mathcal{C}M = \cup_{x\in M}\mathcal{C}_xM$ is a fiber bundle over M such that each fiber $\mathcal{C}_xM$ is an open cone in $T_xM \setminus \{0\}$. Suppose that L has the following properties

(a) L is C^∞ in $\mathcal{C}M$,

(b) the fundamental form g_y is nondegenerate for $y \in \mathcal{C}M$,

then L is called an *singular Finsler metric* and (M, L) is called a *singular Finsler space.*

The Cartan torsion can be defined for singular Finsler metrics exactly in the same way as for the regular ones.

Example 2.1.4 Given an arbitrary function $\mathbf{I}(s)$, $a < s < b$. Let $\phi(s)$ and $\psi(s)$ be two linearly independent solutions of the following equation

$$y''(s) + \mathbf{I}(s)y'(s) + y(s) = 0. \tag{2.20}$$

Assume that $\psi(s) \neq 0$ and $\chi(s) := \frac{\phi(s)}{\psi(s)}$ has the inverse χ^{-1} on an interval (a, b).

Let M be a surface. Suppose that there are two pointwise linearly independent 1-forms α and β on M. Define

$$F(y) := \frac{\alpha(y)}{\phi \circ \chi^{-1}\Big(\frac{\alpha(y)}{\beta(y)}\Big)}, \qquad y \in TM. \tag{2.21}$$

Clearly, F satisfies the homogeneity condition

$$F(\lambda y) = \lambda F(y), \qquad \lambda > 0.$$

Let $\{\mathbf{a}, \mathbf{b}\}$ be a frame on M, which is dual to $\{\alpha, \beta\}$. At each point $x \in M$, let

$$c_x(s) := \phi(s)\mathbf{a}_x + \psi(s)\mathbf{b}_x.$$

c_x is a curve in T_xM. Observe that

$$F_x(c_x(s)) = \frac{\alpha(c_x(s))}{\phi \circ \chi^{-1}\Big(\frac{\alpha(c_x(s))}{\beta(c_x(s))}\Big)} = \frac{\phi(s)}{\phi \circ \chi^{-1}\Big(\frac{\phi(s)}{\psi(s)}\Big)} = 1. \tag{2.22}$$

Thus the indicatrix of F_x in T_xM is the curve c_x. Lemma 1.2.3 tells us that F_x is a singular positive definite Minkowski functional on T_xM with the main scalar $\mathbf{I}(s)$. We obtain a Finsler metric $F = \{F_x\}_{x \in M}$ on M such that the main scalar of F_x in T_xM is the same $\mathbf{I}(s)$ for all $x \in M$. It is known that when $\mathbf{I}(s) = constant$, F is a Berwald metric. This fact is first proved by L. Berwald [Bw5]. ♯

Example 2.1.5 Let m be an odd integer. The following metric is a generalized Berwald-Moòr metric.

$$F(y) := \Big[a_{i_1 \cdots i_m}(x) y^{i_1} \cdots y^{i_m}\Big]^{\frac{1}{m}}, \tag{2.23}$$

where $a_{i_1 \cdots i_m}(x)$ are functions on an open subset $\Omega \subset \mathrm{R}^n$. F is a singular Finsler metric on Ω in many cases. The Finsler metric in (2.23) is also called an *m-th root metric*. In his work on generalized Berwald spaces [Wa1] [Wa2], V.V. Wagner first studied a special class of m-th root metrics. Later on, M. Matsumoto and others wrote several papers on m-th root metrics. See [Ma12], [MaOk] and references therein. ♯

Example 2.1.6 Given $\lambda > 0$, $a, b, c > 0$. Define $L : T\mathrm{R}^2 \to [0, \infty)$ by

$$L := \exp\Big[-2\alpha_1 x + 2(\lambda+1)\alpha_2 x^2 + 2\nu_3 xy\Big]\Big(\frac{v^{1+\frac{1}{\lambda}}}{u^{\frac{1}{\lambda}}}\Big)^2, \tag{2.24}$$

where λ, α_i are positive constants. L is a Berwald metric. It is locally Minkowskian if and only if $\nu_3 = 0$. See Examples 6.1.2 and 6.1.2 below. This Finsler metric has an application to heterochrony [AnHaMo].

There is another interesting metric studied by Antonelli [An1].

$$L := \exp\left[4\frac{a^2+b^2}{a+b}\cdot(ax+by)\right]\exp\left[2\frac{b-a}{b+a}\tan^{-1}\left(\frac{u}{v}\right)\right]\left(u^2+v^2\right). \quad (2.25)$$

This metric is actually a locally Minkowskian metric. See Example 8.2.5 below.

♯

The above mentioned Finsler metrics belong to the following family of singular Finsler metrics on R^2.

$$L = e^{\phi(x,y)}N(u,v), \quad (2.26)$$

where N is a Minkowski functional with constant main scalar. The Finsler metric in (2.26) was used in coral reef ecology [AnInMa]. By [La1][La2][Bw5], we know that F is a Berwald metric.

A Finsler metric L is called an *Antonelli metric* if at every point, there is a local coordinate system (called an *adapted system*) in which $\Gamma^i_{jk}(y) = \Gamma^i_{jk}(y^1, \cdots y^n)$ are functions of (y^i) only. This kind of Finsler metrics were first studied by P.L. Antonelli [An2]. He calls them *y-Berwald metrics*. Antonelli metrics arise in time-sequencing change models in the evolution of colonial systems. Any two-dimensional Antonelli metric is conformal to a locally Minkowskian metric which can be expressed in an adapted coordinate system as follows

$$L = e^{ax+by+c}N(u,v), \quad (2.27)$$

where F_o is a Minkowski functional on R^2. Conversely, any Finsler metric in the form (2.27) must be an Antonelli metric (see Example 4.2.3). This fact was proved in [AnInMa]. Two-dimensional Antonelli metrics were also studied in [AnMaZa].

The simplest Antonelli metrics are those with constant Christoffel symbols in adapted coordinates. They are also called *constant-Berwald metrics*. Constant-Berwald metrics play an important role in the recent development of ecology. Two-dimensional constant-Berwald metrics must be locally projectively flat. In fact, they are either locally Minkowskian or pseudo-Riemannian. See [AnHaMo] [AnMa2] [AnInMa] for further discussion.

Example 2.1.7 ([In]) In Luneburg's theory of binocular vision, the binocular visual space is presented in the form of a three-dimensional Riemannian space of constant negative curvature with the following metric on the ball $\mathbb{B}^3(r)$ of radius $r = 2/\sqrt{-\kappa}$

$$L_1 := \frac{u^2+v^2+w^2}{\left[1+\frac{1}{4}\kappa\left(x^2+y^2+z^2\right)\right]^2}.$$

Note that $4L_1$ is the standard Poincare metric on the unit ball $\mathbb{B}^3(1)$ when $\kappa = -4$. Taking a consideration of other facts, R. Ingarden generalizes Luneburg's metric to

$$L_a := \frac{\Big(u^2+v^2+w^2\Big)^2}{\Big[1+\frac{1}{4}\kappa\Big(x^2+y^2+z^2\Big)\Big]^2\Big(u^2+v^2+a^2v^2\Big)}. \qquad (2.28)$$

Clearly, L_a is conformally Minkowskian. ♯

Some Finsler metrics are constructed from Lagrange metrics. We now briefly discuss Lagrange metrics below. Further discussion will be given in the next chapter on variational problems.

Definition 2.1.2 Let $I = (\alpha, \beta)$ and N be an $(n-1)$-dimensional manifold. A function $\phi : I \times TN \to \mathrm{R}$ is a called a *Lagrange metric* on $I \times N$ if it has the following properties

(i) ϕ is C^∞ on $I \times (TN \setminus \{0\})$,

(ii) for each $\eta \in N$, the restriction $\phi_\eta := \phi|_{I\times T_\eta N}$ is a Lagrange functional on $T_\eta N$ (see Definition 1.1.2)).

Let $\phi = \phi(s, \xi)$ be a Lagrange metric on $M = I \times N$. Define a map $L : TM \to \mathrm{R}$ by

$$L(y) = \big[y^1\phi(s,\xi)\big]^2, \qquad (2.29)$$

where $y = y^1\frac{\partial}{\partial s} \oplus y^1\xi \in T_{(s,\eta)}(I \times N)$. By Lemma 1.1.3, we know that L is a singular Finsler metric on M. It is possible that L is singular in the directions $y = \xi \in T_\eta N \subset T_{(s,\eta)}(I \times N)$. In many problems, however, the singularity does not cause much trouble. We can employ the methods developed in Finsler geometry to study Lagrange metrics.

Example 2.1.8 In irreversible thermodynamics, the Riemannian approximation ignores the asymmetry of relativity entropy (information). If we take a consideration of the ignored items, say, the third term of the Taylor expansion, we obtain a Lagrange metric such as

$$\phi = A_{ab}(s,\eta)\xi^a\xi^b + A_{abc}(s,\eta)\xi^a\xi^b\xi^c, \qquad (2.30)$$

where $a, b, c = 2, \cdots, n$ and $\det(A_{ab}) > 0$. This is studied by R. Ingarden and his collaborators (see [In] and references therein). If a "thermodynamic time" is introduced as a new coordinate, we obtain a Finsler metric in the form

$$L = \Big[\frac{A_{ab}(x)y^ay^b}{y^1} + \frac{A_{abc}(x)y^ay^by^c}{(y^1)^2}\Big]^2, \qquad (2.31)$$

where $a = 2, \cdots, n$. Here $(A_{ab}(x))$ is a symmetric matrix satisfying $\det(A_{ab}(x)) > 0$. If the second term is ignored, $A_{abc} \approx 0$, then L is approximately a Kropina metric. ♯

2.2 Spherical Metrics

In this section, we are going to introduce some important Finsler metrics on S^n. By the maps $\psi_\pm : R^n \to S^n_\pm$ in (2.13), we can express them in R^n.

For $y \in T_x R^n = R^n$, define

$$A_\varepsilon(y) := |x|^2|y|^2 - \langle x, y\rangle^2 + \varepsilon\, |y|^2 + \frac{2(1-\varepsilon^2)\langle x, y\rangle^2}{|x|^4 + 2\varepsilon|x|^2 + 1},$$

$$B_\varepsilon(y) := \Big(|x|^2|y|^2 - \langle x, y\rangle^2\Big)^2 + 2\varepsilon\,\Big(|x|^2|y|^2 - \langle x, y\rangle^2\Big)|y|^2 + |y|^4,$$

where $|\cdot|$ and $\langle\ ,\ \rangle$ denote the canonical Euclidean norm and inner product in R^n respectively. A_ε and B_ε are functions on $TR^n = R^n \times R^n$. For $0 < \varepsilon \le 1$, define

$$F_\varepsilon(y) := \sqrt{\frac{A_\varepsilon(y) + \sqrt{B_\varepsilon(y)}}{2\Big(|x|^4 + 2\varepsilon|x|^2 + 1\Big)}} + \frac{\sqrt{1-\varepsilon^2}\langle x, y\rangle}{|x|^4 + 2\varepsilon\,|x|^2 + 1}, \quad y \in T_x R^n. \tag{2.32}$$

F_ε is a family of Finsler metrics on R^n. Note that $L := (F_1)^2$ is just the Riemannian metric in (2.9).

The family of Finsler metrics, F_ε, have the following important properties:

(i) The geodesics of F_ε are straight lines in R^n as point sets;

(ii) The flag curvature of F_ε is always equal to 1.

We will give the notions of geodesics and flag curvature in later sections. To verify these facts directly, we might have to employ a computer program (such as Maple and Mathematica) on a fast computer.

Define

$$\tilde{F}_\varepsilon(y) := \frac{1}{2}\Big(F_\varepsilon(-y) + F_\varepsilon(y)\Big) = \sqrt{\frac{A_\varepsilon(y) + \sqrt{B_\varepsilon(y)}}{2\Big(|x|^4 + 2\varepsilon|x|^2 + 1\Big)}}. \tag{2.33}$$

$\tilde{F}_\varepsilon$ is reversible, i.e., $\tilde{F}_\varepsilon(-y) = \tilde{F}_\varepsilon(y)$. The geodesics of $\tilde{F}_\varepsilon$ are still straight lines in R^n (see Section 12.2). However, the flag curvature of $\tilde{F}_\varepsilon$ is no longer constant. We only know partial information on the geometric properties of $\tilde{F}_\varepsilon$. The flag curvature of $\tilde{F}_\varepsilon$ is negative near the origin, but always larger than -2. While the flag curvature approaches 1 near infinity.

Let $\psi_\pm : R^n \to S^n_\pm \subset R^{n+1}$ be two diffeomorphisms defined in (2.13). Then ψ_+, ψ_- push the Finsler metric F_ε to a smooth Finsler metric on S^n. The resulting Finsler metrics on S^n are projectively flat Finsler metrics of constant curvature 1. These important examples are constructed recently by R. Bryant [Br3]. Bryant's examples in dimension two are described in the following

Example 2.2.1 ([Br1][Br2]) As we have mentioned above, in dimension two, the family of Finsler metrics in (2.32) are the pull-back of the Bryant metrics on S^2. Bryant defines his metrics in a different way. Let V be a three-dimensional real vector space and $\mathrm{V}\otimes\mathbb{C}$ its complex vector space. Take a basis $\{e_1, e_2, e_3\}$ for V and define a quadratic Q on $\mathrm{V}\times\mathbb{C}$ by

$$Q(u,v) = e^{i\alpha}u^1v^1 + e^{i\beta}u^2v^2 + e^{-i\alpha}u^3v^3, \tag{2.34}$$

where $u = u^ie_i, v = v^ie_i$ and $\alpha, \beta \in \mathrm{R}$. For $y \in \mathrm{V}\setminus\{0\}$, let $[y] := \{ty,\ t > 0\}$. Then $\mathrm{S} := \{[y],\ y \in \mathrm{V}\setminus\{0\}\}$ is diffeomorphic to the standard unit sphere $\mathbb{S}^2$ in the Euclidean space $\mathbb{R}^3$. For a vector $v \in \mathrm{V}$, denote by $[y,v] \in T_{[y]}\mathrm{S}$ the tangent vector to the curve $c(t) := [y+tv]$ at $t = 0$. Note that $[y,v] = [y', v']$ if and only if $y' = ay$ and $v' = av + by$ for some $a > 0$ and $b \in \mathrm{R}$. Define $F : T\mathrm{S} \to \mathrm{R}$ by

$$F([y,v]) := \mathcal{R}\Big[\sqrt{\frac{Q(y,y)Q(v,v) - Q(y,v)^2}{Q(y,y)^2}} - i\frac{Q(y,v)}{Q(y,y)}\Big], \tag{2.35}$$

where $\mathcal{R}[\cdot]$ denotes the real part of a complex number. Clearly, F is well-defined. Assume that $|\beta| \le \alpha < \frac{\pi}{2}$. Then F is indeed a Finsler metric on S^2. Bryant has verified that F has constant curvature $\mathbf{K} = 1$. ♯

There are many other Finsler metrics of positive constant curvature. Recently, D. Bao and Z. Shen discover another family of Finsler metrics on S^{2n+1} with constant flag curvature $\mathbf{K} = 1$. The metrics in three dimension are described in the following example.

Example 2.2.2 (Bao-Shen[BaSh2]) Let $\zeta^1, \zeta^2, \zeta^3$ be the standard right invariant 1-forms on S^3 such that

$$\alpha(y) := \sqrt{[\zeta^1(y)]^2 + [\zeta^2(y)]^2 + [\zeta^3(y)]^2}$$

is the standard Riemannian metric of constant curvature 1. For $k \ge 1$, define

$$\alpha_k(y) := \sqrt{k^2[\zeta^1(y)]^2 + k[\zeta^2(y)]^2 + k[\zeta^3(y)]^2}$$

and

$$\beta_k(y) := \sqrt{k^2 - k}\,\zeta^1.$$

Then

$$F_k := \alpha_k + \beta_k \tag{2.36}$$

is a Finsler metric on S^3. Bao-Shen show that F_k is of constant curvature $\mathbf{K} = 1$ for any $k \ge 1$. Further, this family of Randers metrics are not locally projectively flat.

Let (x, y, z, u, v, w) be the standard coordinate system in $T\mathrm{R}^3 = \mathrm{R}^3 \times \mathrm{R}^3$. Let $\varphi_\pm : \mathrm{R}^3 \to \mathrm{S}^3_\pm$ denote the diffeomorphisms defined in (2.13). We can express

F_κ on R^3. Pulling back ζ^i by φ_ε^{-1} onto R^3, where $\varepsilon = \pm$, we obtain

$$\zeta^1 = \frac{-\varepsilon dx - zdy + ydz}{x^2+y^2+z^2+1} \tag{2.37}$$

$$\zeta^2 = \frac{zdx - \varepsilon dy - xdz}{x^2+y^2+z^2+1} \tag{2.38}$$

$$\zeta^3 = \frac{-ydx + xdy - \varepsilon dz}{x^2+y^2+z^2+1}. \tag{2.39}$$

Plugging them into α_κ and β_κ, we obtain

$$\alpha_k = \frac{\sqrt{k^2(\varepsilon u + zv - yw)^2 + k(zu - \varepsilon v - xw)^2 + k(yu - xv + \varepsilon w)^2}}{1+x^2+y^2+z^2}$$

$$\beta_k = -\sqrt{k^2-k}\,\frac{\varepsilon u + zv - yw}{1+x^2+y^2+z^2}.$$

♯

2.3 Hyperbolic Metrics

Let Ω be a bounded open domain with boundary $\partial\Omega$ in R^n. Suppose that Ω is strictly convex, i.e., any line segment joining two points in Ω is strictly contained in Ω. Then for an arbitrary point $p \in \Omega$, there is an unique functional $\varphi : \mathrm{R}^n \to [0, \infty)$ such that

$$\varphi(\lambda y) = \lambda\varphi(y), \qquad \lambda > 0,\ y \in \mathrm{R}^n$$

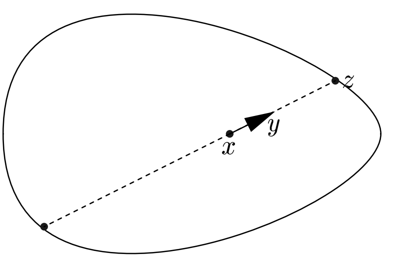
and

$$\varphi(y - p) = 1, \qquad y \in \partial\Omega.$$

Ω or $\partial\Omega$ is said to be *strongly convex* if the above defined functional φ is a Minkowski functional on R^n for some $p \in \Omega$. One can show that the strong convexity of Ω is independent of the choice of a particular point $p \in \Omega$.

Let Ω be a strongly convex domain in a vector space R^n. For $0 \neq y \in T_x\Omega \approx \mathrm{R}^n$, define $F(y) > 0$ by

$$x + \frac{y}{F(y)} = z \in \partial\Omega. \tag{2.40}$$



We obtain a Finsler metrics F on Ω, which is called the *Funk metric* on Ω. Moreover, we have the following lemma due to Okada [Ok].

Lemma 2.3.1 (Okada) *The Funk metric F on a strongly convex domain $\Omega \subset \mathrm{R}^n$ satisfies*

$$F_{x^k} = F F_{y^k}. \tag{2.41}$$

Proof: Since Ω is strongly convex, by assumption, there is a Minkowski functional φ on R^n and a point $p \in \Omega$ such that

$$\varphi(y-p) = 1, \qquad y \in \partial\Omega.$$

Thus

$$\varphi\Big(x + \frac{y}{F(y)} - p\Big) = 1. \tag{2.42}$$

By differentiating (2.42) with respect to x^j or y^j, we obtain

$$\Big(\delta^i_j - F^{-2} F_{x^j} y^i\Big)\varphi_{z^i}(z) = 0 \tag{2.43}$$

$$\Big(\delta^i_j - F^{-1} F_{y^j} y^i\Big)\varphi_{z^i}(z) = 0, \tag{2.44}$$

where $z := x + y/F(y) - p$. It follows from (2.43) and (2.44) that

$$\Big(F_{x^j} - F F_{y^j}\Big)\varphi_{z^i}(z) y^i = 0.$$

Observe that

$$\varphi_{z^i}(z) y^i = \frac{1}{\varphi(z)} g_z(z,y) \neq 0.$$

Thus

$$F_{x^j} - F F_{y^j} = 0.$$

This implies (2.41). Q.E.D.

Let F be the Funk metric on a strongly convex $\Omega \subset \mathrm{R}^n$. Define

$$\tilde{F}(y) := \frac{1}{2}\Big(F(-y) + F(y)\Big). \tag{2.45}$$

$\tilde{F}$ is called the *Klein metric* on Ω [Kl]. Using (2.41), we obtain

$$\tilde{F}_{x^k}(y) y^k = \tilde{F}(y)\Big\{F(y) - F(-y)\Big\}. \tag{2.46}$$

The Funk metric F is positively complete with constant curvature $\mathbf{K} = -1/4$ and the Klein metric $\tilde{F}$ is complete with constant curvature $\mathbf{K} = -1$. All the geodesics of F and $\tilde{F}$ are straight lines. We will prove this fact using (2.41) and (2.46). See Section 4.3 and Theorem 12.2.11 below. See also [Fu1] [Bw3] [Bw4] [Bu3] [BuKe] [Ok] [Za] for related discussion.

The reader can easily verify that the Funk metric F and the Klein metric $\tilde{F}$ on the unit ball B^n in $\mathbb{R}^n$ are given by

$$F = \frac{\sqrt{|y|^2 - \Big(|x|^2|y|^2 - \langle x,y\rangle^2\Big)} + \langle x,y\rangle}{1-|x|^2}, \tag{2.47}$$

$$\tilde{F} = \frac{\sqrt{|y|^2 - \Big(|x|^2|y|^2 - \langle x,y\rangle^2\Big)}}{1-|x|^2}, \tag{2.48}$$

where $\langle , \rangle$ and $|\cdot|$ denote the standard inner product and Euclidean norm in $\mathbb{R}^n$ respectively. Note that $\tilde{F}$ is just the Klein metric in (2.11) on B^n.

Chapter 3

SODEs and Variational Problems

Second order ordinary differential equations (SODEs) arise in many areas of natural science. A special class of SODEs come from the variational problems of Lagrange metrics (including Finsler metrics). In general, it is impossible to find explicit solutions to a system of SODEs. Thus, one would like to know the behavior of the solutions based on some data of the system. In the early twentieth century, many geometers had made great effort to study SODEs using geometric methods. Among them are L. Berwald [Bw1, 1926; Bw7, 1947; Bw8, 1947], T.Y. Thomas [Th1, 1925; Th2, 1926; VeTh2, 1926], O. Veblen [Ve1, 1925; Ve2, 1929], J. Douglas [Dg1, 1928; Dg2, 1941], M. S. Knebelman [Kn, 1929], D. Kosambi [Ko2, 1933; Ko3, 1935], E. Cartan [Ca, 1933] and S.S. Chern [Ch2, 1939], etc. In this chapter, we will show that every system of SODEs (also called a semispray) can be studied via the associated system of homogeneous SODEs (also called a spray), and every Lagrange metric can be studied via the associated Finsler metric. Therefore, in the following chapters, we will be mainly concerned with sprays and Finsler metrics, rather than semisprays and Lagrange metrics.

3.1 SODEs

Let $\mathcal{U} = (a,b) \times \Omega$, where Ω is an open subset in R^{n-1}. Consider the following system of SODEs on $\mathcal{U}$

$$\frac{d^2 f^a}{ds^2} = \Phi^a\Big(s, f, \frac{df}{ds}\Big), \qquad a = 2, \cdots, n, \tag{3.1}$$

where $\Phi : \mathcal{U} \times \mathrm{R}^{n-1} \to \mathrm{R}^{n-1}$ are vector-valued functions. The graph of a solution $f(s) = \big(f^2(s), \cdots, f^n(s)\big)$ is the curve $c(s) := \big(s, f(s)\big)$ in $\mathcal{U}$.

Let

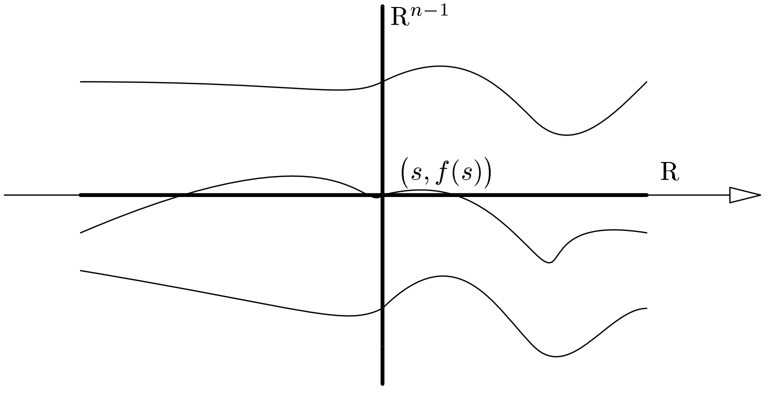
$$\mathcal{S} := \Big(1, \xi^2, \cdots, \xi^n, \Phi^2(s,\eta,\xi), \cdots, \Phi^n(s,\eta,\xi)\Big).$$

$\mathcal{S}$ is a vector field on $\mathcal{U} \times \mathrm{R}^{n-1}$. Let (s, η^a, ξ^a) denote the standard coordinate system in $\mathcal{U} \times \mathrm{R}^{n-1}$. We can express $\mathcal{S}$ in the following form

$$\mathcal{S} = \frac{\partial}{\partial s} + \xi^a \frac{\partial}{\partial \eta^a} + \Phi^a(s, \eta, \xi) \frac{\partial}{\partial \xi^a}.$$

$\mathcal{S}$ is called a *semispray* on $\mathcal{U}$. The notion of semisprays can be extended to fiber bundles $\pi : M \to (a, b)$.

It is easy to see that $f(s)$ is a solution of (3.1) if and only if its lift $\dot{c}(s) := \left(1, f(s), \frac{df}{ds}(s)\right)$ is an integral curve of $\mathcal{S}$ in $\mathcal{U} \times \mathrm{R}^{n-1}$. We will show that every semispray or a system of SODEs can be converted to a spray or a system of homogeneous SODEs in a natural way.



A homogeneous system of SODEs on an open subset $\mathcal{U} \subset \mathrm{R}^n$ is a system as follows

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0, \tag{3.2}$$

where G^i's satisfy

$$G^i(x, \lambda y) = \lambda^2 G^i(x, y), \qquad \lambda > 0. \tag{3.3}$$

Because of the homogeneity of G^i, any solution $x(t) = (x^1(t), \cdots, x^n(t))$ of (3.2) has the following property: for any $\lambda > 0$, $\tilde{x}(t) := x(\lambda t)$ is again a solution of (3.2).

Let

$$\mathbf{G} := \left(y^1, \cdots, y^n, G^1(x, y), \cdots, G^n(x, y)\right).$$

$\mathbf{G}$ is a vector field on $\mathcal{U} \times \mathrm{R}^n$. Let (x^i, y^i) denote the standard coordinate system in $\mathcal{U} \times \mathrm{R}^n$. We can express $\mathbf{G}$ in the following form

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}.$$

$\mathbf{G}$ is called a *spray* on $\mathcal{U}$. The notion of sprays can be extended to manifolds. See Chapter 4 below.

It is easy to see that $x(t)$ is a solution of (3.2) if and only if its lift $\dot{x}(t) := \Big(x(t), \frac{dx}{dt}(t)\Big)$ is an integral curve of $\mathbf{G}$ in $\mathcal{U} \times \mathrm{R}^n$.

Now we assume that two systems (3.1) and (3.2) are related by

$$\Phi^a(s,\eta,\xi) = 2\xi^a G^1\big(s,\eta,1,\xi\big) - 2G^a\big(s,\eta,1,\xi\big). \tag{3.4}$$

Clearly, for a given set of functions Φ^a, there are infinitely many sets of homogeneous functions G^i satisfying (3.4). A special case is when G^i are given by

$$\begin{cases} G^1(x,y) & = 0 \\ G^a(x,y) & = -\frac{1}{2} y^1 y^1 \Phi^a(s,\eta,\xi) \end{cases}$$

where $s = x^1, \eta^a = x^a$ and $\xi^a = y^a/y^1$.

Suppose that $f(s) = (f^2(s), \cdots, f^n(s))$ is a solution of (3.1) and $s = s(t)$ is a solution of the following equation with $\frac{ds}{dt} > 0$

$$\frac{d^2 s}{dt^2} + 2G^1\Big(s, f(s), 1, \frac{df}{ds}(s)\Big)\Big(\frac{ds}{dt}\Big)^2 = 0. \tag{3.5}$$

Let $x^1(t) := s(t)$ and $x^a(t) := f^a(s(t))$, $a = 2, \cdots, n$. It is easy to verify that $x(t) = (x^1(t), \cdots, x^n(t))$ is a solution of (3.2) with $\frac{dx^1}{dt} > 0$.

Conversely, suppose that $x(t) = (x^1(t), \cdots, x^n(t))$ is a solution of (3.2) with $\frac{dx^1}{dt} > 0$. First, we see that $s = x^1(t)$ satisfies (3.5) with $\frac{ds}{dt} > 0$. Note that $s = x^1(t)$ has the inverse $t = t(s)$. Then $f^a(s) := x^a(t(s))$ satisfy

$$\begin{aligned} \frac{d^2 f^a}{ds^2} &= \frac{\frac{d^2 x^a}{dt^2}\frac{ds}{dt} - \frac{dx^a}{dt}\frac{d^2 s}{dt^2}}{\Big(\frac{ds}{dt}\Big)^3} \\ &= \frac{-2G^a\Big(x, \frac{dx}{dt}\Big)\frac{ds}{dt} + 2\frac{dx^a}{dt}G^1\Big(x, \frac{dx}{dt}\Big)}{\Big(\frac{ds}{dt}\Big)^3} \\ &= 2\frac{df^a}{ds}G^1\Big(s, f, 1, \frac{df}{ds}\Big) - 2G^a\Big(s, f, 1, \frac{df}{ds}\Big) \\ &= \Phi^a\Big(s, f, \frac{df}{ds}\Big). \end{aligned}$$

Thus $f(s) = (f^2(s), \cdots, f^n(s))$ is a solution of (3.1). This proves (a) in the following lemma. The proof of (b) follows the same argument, so is omitted.

Lemma 3.1.1 *Let $\Phi^a = \Phi^a(s,\eta,\xi)$ be a set of functions in (3.1) and $G^i = G^i(x,y)$ a set of homogeneous functions in (3.2).*

(a) Suppose that Φ^a and G^i are related by (3.4). Then for any solution $x(t) = (x^1(t), \cdots, x^n(t))$ of (3.2) with $\frac{dx^1}{dt} > 0$, $f(s) := \big(x^2(t), \cdots, x^n(t)\big)$ is a

solution of (3.1),where $s = x^1(t)$ satisfies (3.5) with $\frac{ds}{dt} > 0$. Conversely, for any solution $f(s) = (f^2(s), \cdots, f^n(s))$ of (3.1) and any solution $s = s(t)$ of (3.5) with $\frac{ds}{dt} > 0$, $x(t) := (s, f^2(s), \cdots, f^n(s))$ is a solution of (3.2) with $\frac{dx^1}{dt} > 0$.

(b) Suppose that the solutions of (3.1) and (3.2) are related as in (a). Then Φ^a and G^i are related by (3.4).

Therefore, we can study a system of SODEs (3.1) by investigating a system of homogeneous SODEs (3.2) provided that Φ^a and G^i are replaced by (3.4). Since there are infinitely many systems (3.2) satisfying (3.4) for a given system (3.1), we have to study the properties of (3.1) which are independent of the choice of a particular set of functions G^i satisfying (3.4). We will discuss this problem in Chapter 12.

Example 3.1.1 Consider a system of SODEs on an open subset $\mathcal{U} = (a,b) \times \Omega \subset \mathrm{R}^n$.

$$\frac{d^2 f^a}{ds^2} = \Phi^a\Big(s, f, \frac{df}{ds}\Big), \qquad a = 2, \cdots, n, \tag{3.6}$$

where

$$\Phi^a(s,\eta,\xi) := A^a(s,\eta) + B^a_b(s,\eta)\xi^b + C^a_{bc}(s,\eta)\xi^b\xi^c + D_{bc}(s,\eta)\xi^b\xi^c\, \xi^a. \tag{3.7}$$

Define

$$\begin{cases} G^1(x,y) = \frac{1}{2} D_{bc}(x) y^b y^c, \\ G^a(x,y) = -\frac{1}{2}\Big(A^a(x)y^1y^1 + B^a_b(x)y^1y^b + C^a_{bc}(x)y^by^c\Big). \end{cases} \tag{3.8}$$

Then Φ^a and G^i satisfy (3.4). By Lemma 3.1.1, we can study (3.6) by investigating the following homogeneous system with regular coefficients

$$\begin{cases} \frac{d^2x^1}{dt^2} + D_{bc}(x)\frac{dx^b}{dt}\frac{dx^c}{dt} = 0 \\ \frac{d^2x^a}{dt^2} - A^a(x)\frac{dx^1}{dt}\frac{dx^1}{dt} - B^a_b(x)\frac{dx^1}{dt}\frac{dx^b}{dt} - C^a_{bc}(x)\frac{dx^b}{dt}\frac{dx^c}{dt} = 0. \end{cases} \tag{3.9}$$

For a single SODE on an open subset $\mathcal{U} = (a,b) \times (c,d) \subset \mathrm{R}^2$, we usually express it as follows

$$\frac{d^2y}{dx^2} = \Phi\Big(x, y, \frac{dy}{dx}\Big), \tag{3.10}$$

where $\Phi = \Phi(x,y,\xi)$ is a C^∞ function on $\mathcal{U} \times \mathrm{R}$. Assume that Φ is in the following form

$$\Phi = A(x,y) + B(x,y)\xi + C(x,y)\xi^2 + D(x,y)\xi^3. \tag{3.11}$$

In this case, we can construct several homogeneous systems

$$\begin{cases} \frac{d^2x}{dt^2} + 2G\Big(x, y, \frac{dx}{dt}, \frac{dy}{dt}\Big) = 0 \\ \frac{d^2y}{dt^2} + 2H\Big(x, y, \frac{dx}{dt}, \frac{dy}{dt}\Big) = 0 \end{cases} \tag{3.12}$$

such that G and H satisfy (3.4), namely,

$$\Phi(x,y,\xi) = 2\xi G(x,y,1,\xi) - 2H(x,y,1,\xi). \tag{3.13}$$

Option 1:

$$\begin{cases} G &= \frac{1}{2}D(x,y)v^2 \\ H &= -\frac{1}{2}\Big(A(x,y)u^2 + B(x,y)uv + C(x,y)v^2\Big) \end{cases} \tag{3.14}$$

Option 2:

$$\begin{cases} G &= \frac{1}{2}\Big(B(x,y)u^2 + C(x,y)uv + D(x,y)v^2 \\ H &= -\frac{1}{2}A(x,y)u^2 \end{cases} \tag{3.15}$$

Option 3:

$$\begin{cases} G &= \frac{1}{6}B(x,y)u^2 + \frac{1}{3}C(x,y)uv + \frac{1}{2}D(x,y)v^2 \\ H &= -\frac{1}{2}A(x,y)u^2 - \frac{1}{3}B(x,y)uv - \frac{1}{6}C(x,y)v^2 \end{cases} \tag{3.16}$$

Notice that (3.13) is satisfied by any set of G and H in (3.14)-(3.16). Thus, by Lemma 3.1.1, we can study (3.6) by investigating either homogeneous system. When $D = 0$, we have more choices for regular systems whose coefficients G and H satisfy (3.13). ♯

Some systems of first order ordinary differential equations can also be studied by converting them to systems of SODEs.

Example 3.1.2 (Volterra-Hamilton System with Discounting [AnBu]) E. Baake and V. Křivan independently suggest that a reasonable model for discounted production in ecology can be written as

$$\begin{cases} \frac{dx^a}{dt} = k_{(a)}y^a - h_{(a)}x^a, \qquad \text{(not summed)} \\ \frac{dy^a}{dt} = \gamma^a_{bc}(t,x)y^b y^c + \gamma^a_b(t,x)y^b + e^a(t), \end{cases} \tag{3.17}$$

where $a,b,c = 2,\cdots,n$ and $k_{(a)} \neq 0, h_{(a)}$ are the constants. Take the following transformation

$$\begin{cases} s = t \\ f^a = \exp(h_{(a)}t)x^a \\ g^a = k_{(a)}\exp(h_{(a)}t)y^a. \end{cases} \tag{3.18}$$

The system (3.17) becomes

$$\begin{cases} \frac{df^a}{ds} = g^a \\ \frac{dg^a}{ds} = \Gamma^a_{bc}(s,f)g^b g^c + \Gamma^a_b(s,f)g^b + \Gamma^a(s,f), \end{cases} \tag{3.19}$$

where

$$\begin{aligned} \Gamma^a_{bc} &= \frac{k_{(a)}}{k_{(b)}k_{(c)}}\exp\big[(h_{(a)} - h_{(b)} - h_{(c)})s\big]\gamma^a_{bc}(s,x) \\ \Gamma^a_b &= h_{(a)}\delta^a_b + \frac{k_{(a)}}{k_{(b)}}\exp\big[(h_{(a)} - h_{(b)})s\big]\gamma^a_b(s,x) \\ \Gamma^a &= k_{(a)}\exp(h_{(a)}s)\cdot e^a(s). \end{aligned}$$

where $x^a = \exp[-h_{(a)}s]\eta^a$, $a = 2, \cdots, n$.

Set

$$\Phi^a(s,\eta,\xi) := \Gamma^a_{bc}(s,\eta)\xi^b\xi^c + \Gamma^a_b(s,\eta)\xi^b + \Gamma^a(s,\eta).$$

The system (3.19) becomes

$$\frac{d^2 f^a}{ds^2} = \Phi^a\Big(s, f, \frac{df}{ds}\Big). \tag{3.20}$$

See [AnBr] for the systematic study on Volterra-Hamilton models in Ecology. ♯

There are many systems of first order ordinary differential equations which give rise to a system of SODEs. For example, the Lorenz system

$$\begin{cases} \frac{dx}{dt} & = \sigma(y-x) \\ \frac{dy}{dt} & = rx - y - xz \\ \frac{dz}{dt} & = xy - bz \end{cases} \tag{3.21}$$

is equivalent to the following SODE

$$\begin{aligned} \frac{d^2y}{dx^2} &= -\varepsilon xy^4 + \beta x^3y^4 + \beta x^2y^3 + \gamma y^3 - \delta x^{-1}y^2 \\ &\quad + (x^{-1} - \alpha y)\frac{dy}{dx} + 3y^{-1}\Big(\frac{dy}{dx}\Big)^2, \end{aligned} \tag{3.22}$$

where

$$\alpha = \frac{b+\sigma+1}{\sigma}, \quad \beta = \frac{1}{\sigma^2}, \quad \gamma = \frac{b(\sigma+1)}{\sigma^2}, \quad \delta = \frac{\sigma+1}{\sigma}, \quad \varepsilon = \frac{b(r-1)}{\sigma^2}.$$

See [Dr1]-[Dr3] for related discussion and references.

3.2 Variational Problems

In this section, we will discuss the variational problems of Finsler metrics and Lagrange metrics and derive the Euler-Lagrange equations for extremals. In the next section, we will show that any Lagrange metric ϕ can be transfered to a (possibly singular) Finsler metric L such that their Euler-Lagrange equations are related by (3.4).

Variational Problem I: Let $\mathcal{U} = (\alpha,\beta)\times\Omega \subset \mathrm{R}^n$ be an open subset. Consider a C^∞ function $\phi = \phi(s,\eta,\xi)$ on $\mathcal{U}\times\mathrm{R}^{n-1}$. Let

$$h_{ab} := \frac{1}{2}\phi_{\xi^a\xi^b}(s,\eta,\xi).$$

Assume that

$$\det\Big(h_{ab}\Big) \neq 0. \tag{3.23}$$

By definition, ϕ is a *Lagrange metric* on $\mathcal{U}$.

Consider the following variational problem of ϕ on $\mathcal{U}$:

$$\mathcal{E}(f) = \int \phi\Big(s, f(s), \frac{df}{ds}(s)\Big) ds = \text{Extremum}. \tag{3.24}$$

Let $f : [a, b] \to \Omega \subset \mathrm{R}^{n-1}$ be an arbitrary C^∞ function. Take a variation of $H(u, s)$ of f with

$$H(0, s) = f(s), \qquad H(u, a) = f(a), \qquad H(u, b) = f(b).$$

The energy functional

$$\mathcal{E}(u) := \int_a^b \phi\Big(s, H(u, s), \frac{\partial H}{\partial s}(u, s)\Big) ds$$

satisfies

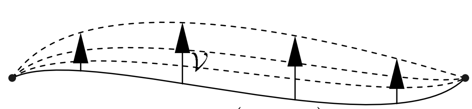
$$\begin{aligned} \mathcal{E}'(u) &= \int_a^b \Big\{ \phi_{\eta^a} \frac{\partial H^a}{\partial u} + \phi_{\xi^a} \frac{\partial^2 H^a}{\partial s \partial u} \Big\} ds \\ &= \int_a^b \Big\{ \phi_{\eta^a} - \frac{d}{ds}\Big(\phi_{\xi^a}\Big) \Big\} \frac{\partial H^a}{\partial u} ds. \end{aligned}$$

Let

$$\mathcal{V}(s) := \frac{\partial H}{\partial u}(0, s).$$

$\mathcal{V}(s)$ is a vector field along $c(s) = (s, f(s))$, $a \le s \le b$. We obtain

$$\mathcal{E}'(0) = \int_a^b \Big\{ \phi_{\eta^a} - \frac{d}{ds}\Big[\phi_{\xi^a}\Big] \Big\} \mathcal{V}^a ds. \tag{3.25}$$



$c(s) = \big(s, f(s)\big)$

Assume that $f(s)$ is an extremal of (3.24). Then $\mathcal{E}'(0) = 0$ for any variation of f. This gives rise to the following system of Euler-Lagrange equations

$$\phi_{\eta^a} - \frac{d}{ds}\Big(\phi_{\xi^a}\Big) = 0, \qquad a = 2, \cdots, n. \tag{3.26}$$

Let $\Big(h^{ab}\Big) := \Big(h_{ab}\Big)^{-1}$. Expand (3.26) as follows

$$\phi_{\eta^a} - \phi_{s\xi^a} - \phi_{\eta^b \xi^a} \frac{df^b}{ds} - 2h_{ab} \frac{d^2 f^b}{ds^2} = 0, \qquad a = 2, \cdots, n.$$

We obtain

$$\frac{d^2 f^a}{ds^2} = \Phi^a\Big(s, f, \frac{df}{ds}\Big), \tag{3.27}$$

where

$$\Phi^a(s,\eta,\xi) := \frac{1}{2}h^{ab}\Big\{\phi_{\eta^b} - \phi_{s\xi^b} - \phi_{\eta^c\xi^b}\xi^c\Big\}. \tag{3.28}$$

Variational Problem II. Let $\mathcal{U}$ be an open domain in R^n. Consider a function $L = L(x,y)$ on $\mathcal{U} \times \mathrm{R}^n$ satisfying

(i) L is C^∞ on $\mathcal{U} \times (\mathrm{R}^n \setminus \{0\})$,

(iia) L is positively homogeneous of degree two, i.e.,

$$L(x,\lambda y) = \lambda^2 L(x,y), \qquad \lambda > 0,\ (x,y) \in \mathcal{U} \times \mathrm{R}^n.$$

(iib) for any $y \neq 0$, the y-Hessian $g_{ij}(x,y)$ of L is nondegenerate , where

$$g_{ij}(x,y) := \frac{1}{2}L_{y^iy^j}(x,y).$$

By definition, L is a *Finsler metric* on $\mathcal{U}$.

Consider the following variational problem of L on $\mathcal{U}$:

$$\mathcal{L}(x) := \int L\Big(x(t), \frac{dx}{dt}(t)\Big)dt = \text{Extremum}. \tag{3.29}$$

Assume that $x(t)$ is an extremal of (3.29). By a similar argument, one obtains the following system of Euler-Lagrange equations

$$L_{x^i} - \frac{d}{dt}\Big(L_{y^i}\Big) = 0, \qquad i = 1,\cdots,n. \tag{3.30}$$

Observe that

$$L_{x^i} - \frac{d}{dt}\Big(L_{y^i}\Big) = -2g_{ij}\Big\{\frac{d^2x^j}{dt^2} + 2G^j\Big\},$$

where

$$G^j(x,y) := \frac{1}{4}g^{jl}\Big\{L_{y^lx^k}y^k - L_{x^l}\Big\} \tag{3.31}$$

We obtain

$$\frac{d^2x^i}{dt^2} + 2G^i\Big(x, \frac{dx}{dt}\Big) = 0. \tag{3.32}$$

Lemma 3.2.1 *For any solution $x(t)$ of (3.30),*

$$L\Big(x(t), \frac{dx}{dt}(t)\Big) = \mathit{constant}. \tag{3.33}$$

Proof: Expand (3.30) as follows

$$L_{x^i} - L_{y^i x^j}\frac{dx^j}{dt} - 2g_{ij}\frac{d^2x^j}{dt^2} = 0. \tag{3.34}$$

Using (3.34), we obtain

$$\begin{aligned}
\frac{d}{dt}\Big[L\Big(x, \frac{dx}{dt}\Big)\Big] &= L_{x^i}\frac{dx^i}{dt} + L_{y^j}\frac{d^2x^j}{dt^2}\\
&= L_{x^i}\frac{dx^i}{dt} + 2g_{ij}\frac{d^2x^j}{dt^2}\frac{dx^i}{dt}\\
&= L_{x^i}\frac{dx^i}{dt} + \Big[L_{x^i} - L_{y^i x^j}\frac{dx^j}{dt}\Big]\frac{dx^i}{dt}\\
&= 2L_{x^i}\frac{dx^i}{dt} - L_{y^i x^j}\frac{dx^i}{dt}\frac{dx^j}{dt}\\
&= \Big[2L_{x^j} - L_{y^i x^j}\frac{dx^i}{dt}\Big]\frac{dx^j}{dt}\\
&= \Big[2L_{x^j} - 2L_{x^j}\Big]\frac{dx^j}{dt} = 0.
\end{aligned}$$

The homogeneity of $L(x, y)$ has been used in the above argument. Q.E.D.

3.3 Relationship Between Euler-Lagrange Equations

In the previous section, we studied the variational problems of a Lagrange metric $\phi = \phi(s, \eta, \xi)$ and a Finsler metric $L = L(x, y)$ on an open subset $\mathcal{U} = (a, b)\times\Omega$ in R^n. We obtained the Euler-Lagrange equations for each of them, namely, we obtained (3.27) for ϕ

$$\Phi^a := \frac{1}{2}h^{ab}\Big\{\phi_{\eta^b} - \phi_{s\xi^b} - \phi_{\eta^c\xi^b}\xi^c\Big\},$$

and (3.32) for L

$$G^i := \frac{1}{4}g^{il}\Big\{L_{y^l x^k}y^k - L_{x^l}\Big\},$$

where

$$h_{ab} := \frac{1}{2}\phi_{\xi^a\xi^b}(s, \eta, \xi), \qquad g_{ij} := \frac{1}{2}L_{y^iy^j}(x, y).$$

In this section, we will show that if ϕ and L are related by

$$L(x, y) := \Big[y^1\phi(s, \eta, \xi)\Big]^2, \tag{3.35}$$

where $x = (s, \eta)$ and $y = y^1(1, \xi)$. Then Φ^a and G^i satisfy

$$\Phi^a(s, \eta, \xi) = 2\xi^a G^1(s, \eta, 1, \xi) - 2G^a(s, \eta, 1, \xi). \tag{3.36}$$

Proposition 3.3.1 *If a Lagrange metric ϕ and a Finsler metric L are related by (3.35), the corresponding functions Φ^a and G^i satisfy(3.36).*

Proof: The system of Euler-Lagrange equations of ϕ are given by

$$\phi_{\eta^a} - \frac{d}{ds}\Big(\phi_{\xi^a}\Big) = 0, \qquad a = 2, \cdots, n. \tag{3.37}$$

The system of Euler-Lagrange equations are given by

$$L_{x^i} - \frac{d}{dt}\Big(L_{y^i}\Big) = 0, \qquad i = 1, \cdots, n. \tag{3.38}$$

By Lemma 3.1.1(b), it suffices to prove that the solutions of (3.37) and (3.38) are related as in Lemma 3.1.1(a).

Assume that $x(t) = (x^1(t), \cdots, x^n(t))$ is a solution of (3.38) with $\frac{dx^1}{dt} > 0$. Let $s = x^1(t)$. Note that $\frac{ds}{dt} > 0$. Thus the inverse function $t = t(s)$ exists. Let $f(s) = (x^2(t(s)), \cdots, x^n(t(s)))$. By Lemma 3.2.1,

$$L\Big(x(t), \frac{dx}{dt}(t)\Big) = \Big[\frac{ds}{dt}\phi\Big(s, f(s), \frac{df}{ds}(s)\Big)\Big]^2 = constant.$$

For a generic solution $x(t)$, we may assume that $L\Big(x(t), \frac{dx}{dt}(t)\Big) \neq 0$. Then

$$\frac{ds}{dt}\phi\Big(s, f(s), \frac{df}{ds}(s)\Big) = constant \neq 0. \tag{3.39}$$

Let

$$F(x, y) := y^1\phi(s, \eta, \xi),$$

where $x = (s, \eta)$ and $y = y^1(1, \xi)$. (3.39) implies that

$$F\Big(x(t), \frac{dx}{dt}(t)\Big) = constant \neq 0.$$

It follows from (3.38) that $x(t)$ satisfies the following equation

$$F_{x^i} - \frac{d}{dt}\Big(F_{y^i}\Big) = 0, \qquad i = 1, \cdots, n. \tag{3.40}$$

Observe that

$$\frac{dx^1}{dt}\Big[\phi_{\eta^a} - \frac{d}{ds}\Big(\phi_{\xi^a}\Big)\Big] = \frac{ds}{dt}\phi_{\eta^a} - \frac{d}{dt}\Big(\phi_{\xi^a}\Big) = F_{x^a} - \frac{d}{dt}\Big(F_{y^a}\Big) = 0.$$

Thus $f(s)$ satisfies (3.37).

Assume that $f(s) = (f^2(s), \cdots, f^n(s))$ is a solution of (3.37) and $s = s(t)$ satisfies (3.39) with $\frac{ds}{dt} > 0$. Let $x(t) := (s(t), f^2(s(t)), \cdots, f^n(s(t)))$. Observe that

$$F_{x^1} - \frac{d}{dt}\Big(F_{y^1}\Big) = -\Big[\phi_{\eta^a} - \frac{d}{ds}\Big(\phi_{\xi^a}\Big)\Big]\frac{df^a}{ds}\frac{ds}{dt} = 0$$

and

$$F_{x^a} - \frac{d}{dt}\Big(F_{y^a}\Big) = \Big[\phi_{\eta^a} - \frac{d}{ds}\Big(\phi_{\xi^a}\Big)\Big]\frac{ds}{dt} = 0.$$

Thus $x(t)$ satisfies (3.40). It must also satisfy (3.38) with $\frac{dx^1}{dt} > 0$. Q.E.D.

Example 3.3.1 Let $\phi = \phi(s,\eta,\xi)$ be a Lagrange metric on $\mathcal{U} = (\alpha,\beta)\times\Omega \subset \mathrm{R}^n$. By definition,

$$\det\Big(h_{ab}\Big) \neq 0,$$

where $h_{ab} := \frac{1}{2}\phi_{\xi^a\xi^b}(s,\eta,\xi)$. Let $\beta = A_a(s,\eta)\xi^a$ be a family of 1-forms on Ω and $A = A(s,\eta)$ a family of scalar functions on Ω. Define a map $L : \mathcal{U}\times(\mathrm{R}^n\setminus\{y^1 = 0\}) \to \mathrm{R}$ by

$$L(x,y) := \Big[y^1\Big(A(s,\eta) + A_a(s,\eta)\xi^a + \phi(s,\eta,\xi)\Big)\Big]^2, \tag{3.41}$$

where $(s,\eta) = (x^1,\cdots,x^n)$ and $\xi = (y^2/y^1,\cdots,y^n/y^1)$. Then L is a singular Finsler metric on $\mathcal{U}$. In particular, if

$$\phi = h_{ab}(s,\eta)\xi^a\xi^b,$$

where $h_{ab} = h_{ba}$ and $\det(h_{ab}) \neq 0$, then

$$L(x,y) = \Big[\frac{A(s,\eta)y^1y^1 + A_a(s,\eta)y^1y^a + h_{ab}(s,\eta)y^ay^b}{y^1}\Big]^2,$$

where $(s,\eta) = (x^1,\cdots,x^n)$. L is a Kropina metric on $\mathcal{U}$. ♯

Example 3.3.2 Consider the following Lagrange metric on an open subset $\mathcal{U} = (\alpha,\beta)\times(\lambda,\mu) \subset \mathrm{R}^2$:

$$\phi = a(x,y)\xi^2 + b(x,y)\xi + c(x,y), \tag{3.42}$$

where $a = a(x,y) \neq 0$. The Euler-Lagrange equation of ϕ is given by

$$\frac{d^2y}{dx^2} = -\frac{1}{2a}\Big[a_y\Big(\frac{dy}{dx}\Big)^2 + 2a_x\frac{dy}{dx} + b_x - c_y\Big]. \tag{3.43}$$

The Finsler metric L associated with ϕ is given by

$$L = \Big[a(x,y)\frac{v^2}{u} + b(x,y)v + c(x,y)u\Big]^2. \tag{3.44}$$

Here we denote by (x,y,u,v) the standard coordinate system of $T\mathcal{U} = \mathcal{U}\times\mathrm{R}^2$. Observe that

$$L_{vv} = \frac{12}{u^2}\Big(a^2v^2 + abuv + \frac{2ac+b^2}{6}u^2\Big),$$

$$L_{uu}L_{vv} - L_{uv}L_{uv} = \frac{8a}{u^6}\Big(av^2 + buv + cu^2\Big)^3.$$

Assume that

$$b^2 - 4ac < 0. \tag{3.45}$$

Then $L_{uu}L_{vv} - L_{uv}L_{uv} > 0$ for $u \neq 0$, regardless the sign of a. Note that

$$(ab)^2 - 4a^2\Big(\frac{2ac + b^2}{6}\Big) = \frac{a^2}{3}\Big(b^2 - 4ac\Big) < 0.$$

Thus $L_{vv} > 0$ for $u \neq 0$. This implies that if (3.45) is satisfied, L is a positive definite Finsler metric for $u \neq 0$.

Given arbitrary C^∞ functions R, P and Q on an open interval I, let

$$\begin{aligned} a &= e^{\int P(x)dx}, \\ b &= E_y(x,y), \\ c &= \Big(2R(x)y - Q(x)y^2\Big)e^{\int P(x)dx} + E_x(x,y), \end{aligned}$$

where $E(x,y)$ is an arbitrary C^∞ function on an open subset $I \times \Omega \subset \mathrm{R}^2$. Then (3.43) simplifies to

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x) \tag{3.46}$$

and the corresponding Finsler metric L is given by

$$L = \Big\{\frac{e^{\int P(x)dx}}{u}\Big[\Big(2R(x)y - Q(x)y^2\Big)u^2 + v^2\Big] + E_x(x,y)u + E_y(x,y)v\Big\}^2. \tag{3.47}$$

The Finsler metric L in (3.47) for (3.46) was constructed by M. Matsumoto [Ma11]. According to the above argument, L is positive definite for $u \neq 0$ on the domain $\Omega \subset \mathrm{R}^2$ if

$$E_y(x,y)^2 - 4e^{\int P(x)dx}\Big[\Big(2R(x)y - Q(x)y^2\Big)e^{\int P(x)dx} + E_x(x,y)\Big] < 0.$$

We will see that this family of metrics are locally projectively flat. See Example 13.6.1. ♯

Chapter 4

Spray Spaces

In this chapter, we will introduce an important geometric structure on a manifold and discuss some basic properties. Roughly speaking, a spray on a manifold M is a family of compatible systems of $2^{\rm nd}$ order ordinary differential equations in local coordinates

$$\ddot{c}^i + 2G^i(\dot{c}) = 0, \tag{4.1}$$

where $(c^i(t))$ denotes the coordinates of a curve $c(t)$, and $G^i(y)$ are positively homogeneous functions of degree two, i.e.,

$$G^i(\lambda y) = \lambda^2 G^i(y), \qquad \lambda > 0.$$

The compatibility gives rise to a globally defined C^∞ vector field $\mathbf{G}$ on the tangent bundle $TM \setminus \{0\}$ which is expressed in local coordinates as follows

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}. \tag{4.2}$$

Conversely, given a globally defined C^∞ vector field $\mathbf{G}$ on $TM \setminus \{0\}$ in the form (4.2) with positively homogeneous coefficients $G^i(y)$ of degree two, one obtains a family of compatible systems of $2^{\rm nd}$ order ordinary differential equations (4.1) in local coordinate system.

4.1 Sprays

First, let us define a spray precisely.

Definition 4.1.1 Let M be a manifold. A *spray* on M is a smooth vector field $\mathbf{G}$ on $TM \setminus \{0\}$ expressed in a standard local coordinate system (x^i, y^i) in TM as follows

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}, \tag{4.3}$$

where $G^i(y)$ are local functions on TM satisfying

$$G^i(\lambda y) = \lambda^2 G^i(y), \qquad \lambda > 0. \tag{4.4}$$

A manifold with a spray is called a *spray space*.

One is referred to [Lan] for some discussion on sprays.

If $\hat{c}$ is an integral curve of $\mathbf{G}$, i.e.,

$$\frac{d\hat{c}}{dt} = \mathbf{G}_{\hat{c}}, \tag{4.5}$$

then the projection $c = \pi \circ \hat{c}$ satisfies

$$\ddot{c}^i + 2G^i(\dot{c}) = 0. \tag{4.6}$$

Conversely, if c satisfies (4.6), then its canonical lift $\hat{c} := \dot{c}$ is an integral curve of $\mathbf{G}$.

Definition 4.1.2 A regular curve c in M is called a *geodesic* of $\mathbf{G}$ if it is the projection of an integral curve of $\mathbf{G}$. We shall call G^i the *spray coefficients* of $\mathbf{G}$.

Example 4.1.1 Let V be an n-dimensional vector space. Let (x^i, y^i) denote a standard local coordinate system for $T\mathrm{V} \approx \mathrm{V} \times \mathrm{V}$ determined by a basis $\{e_i\}_{i=1}^n$ for V. Then

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i}$$

is a well-defined vector field on $T\mathrm{V}$. The geodesics of $\mathbf{G}$ are straight lines in V. $\mathbf{G}$ is called the *standard flat spray* on V. Spray spaces can be viewed as generalized vector spaces. ♯

Example 4.1.2 Consider the following spray on R^3

$$\mathbf{G} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} - 2(xu^2 + zw^2)\frac{\partial}{\partial v}.$$

A curve $c(t) = (x(t), y(t), z(t))$ is a geodesic of $\mathbf{G}$ if it satisfies

$$\begin{aligned}
\frac{d^2x}{dt^2} &= 0, \\
\frac{d^2y}{dt^2} &= -2x\Big(\frac{dx}{dt}\Big)^2 - 2z\Big(\frac{dz}{dt}\Big)^2, \\
\frac{d^2z}{dt^2} &= 0.
\end{aligned}$$

We obtain

$$c(t) = (a,b,c) + (u,v,w)t + \varphi(t)(0,1,0),$$

where

$$\varphi(t) = -\frac{1}{3}\Big(u^3 + w^3\Big)t^3 - \Big(au^2 + cw^2\Big)t^2.$$

Thus the geodesics of **G** are planar curves in R^3. ♯

Example 4.1.3 Consider the following spray on R^2

$$\mathbf{G} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - \frac{v}{r}\sqrt{u^2+v^2}\frac{\partial}{\partial u} + \frac{u}{r}\sqrt{u^2+v^2}\frac{\partial}{\partial v}, \tag{4.7}$$

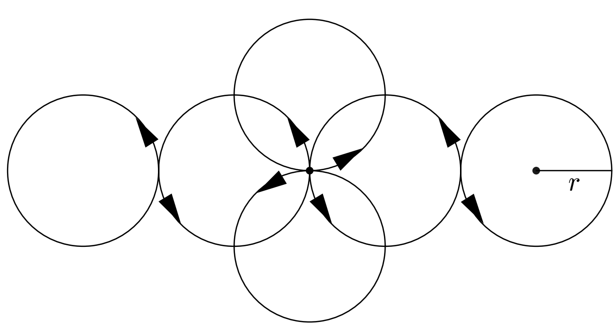
where $r > 0$. The geodesic $c(t) = (x(t), y(t))$ of **G** satisfies

$$\frac{d^2x}{dt^2} + \frac{1}{r}\sqrt{\Big(\frac{dx}{dt}\Big)^2 + \Big(\frac{dy}{dt}\Big)^2}\frac{dy}{dt} = 0,$$
$$\frac{d^2y}{dt^2} - \frac{1}{r}\sqrt{\Big(\frac{dx}{dt}\Big)^2 + \Big(\frac{dy}{dt}\Big)^2}\frac{dx}{dt} = 0.$$

We obtain

$$\frac{dx}{dt} = -\lambda\sin\Big(\frac{\lambda}{r}t + \theta\Big), \qquad \frac{dy}{dt} = \lambda\cos\Big(\frac{\lambda}{r}t + \theta\Big), \tag{4.8}$$

where $\lambda > 0$.



Integrating $\frac{dx}{dt}$ and $\frac{dy}{dt}$ yields

$$x(t) = r\cos\Big(\frac{\lambda}{r}t + \theta\Big) - r\cos\theta + x(0),$$
$$y(t) = r\sin\Big(\frac{\lambda}{r}t + \theta\Big) - r\sin\theta + y(0).$$

Thus the geodesics of **G** are circles of radius r. ♯

Below we are going to look at sprays from different points of view.

I. Horizontal Tangent Bundle Over the Slit Tangent Bundle. Let M be an n-dimensional manifold and $TM \setminus \{0\}$ denote the slit tangent bundle. Take a standard local coordinate system (x^i, y^i) in TM. Let

$$\mathcal{V}TM := \mathrm{span}\Big\{ \frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^n} \Big\}. \tag{4.9}$$

$\mathcal{V}TM$ is a well-defined subbundle of $T(TM \setminus \{0\})$. We call $\mathcal{V}TM$ the *vertical tangent bundle* over $TM \setminus \{0\}$. There is a canonical vector field on $TM \setminus \{0\}$ defined by

$$\mathbf{Y} := y^i \frac{\partial}{\partial y^i}. \tag{4.10}$$

$\mathbf{Y}$ is called the *vertical radial field* which is a section of $\mathcal{V}TM$.

Given a spray on M,

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(y) \frac{\partial}{\partial y^i}.$$

Let

$$N^i_j(y) := \frac{\partial G^i}{\partial y^j}(y). \tag{4.11}$$

We call N^i_j the *connection coefficients* of $\mathbf{G}$. The homogeneity condition (4.4) implies

$$N^i_j(\lambda y) = \lambda N^i_j(y), \qquad \forall \lambda > 0, \tag{4.12}$$

and

$$G^i(y) = \frac{1}{2} N^i_j(y) y^j. \tag{4.13}$$

Let

$$\mathcal{H}TM := \mathrm{span}\Big\{ \frac{\delta}{\delta x^1}, \cdots, \frac{\delta}{\delta x^n} \Big\}, \tag{4.14}$$

where

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^j_i(y) \frac{\partial}{\partial y^j}. \tag{4.15}$$

$\mathcal{H}TM$ is a well-defined subbundle of $T(TM \setminus \{0\})$. We call $\mathcal{H}TM$ the *horizontal tangent bundle* over $TM \setminus \{0\}$. It follows from (4.13) that

$$\mathbf{G} = y^i \frac{\delta}{\delta x^i}. \tag{4.16}$$

Therefore $\mathbf{G}$ is a horizontal vector field which is a section of $\mathcal{H}TM$.

The above argument gives a direct decomposition of the tangent bundle of $TM \setminus \{0\}$.

$$T(TM \setminus \{0\}) = \mathcal{V}TM \oplus \mathcal{H}TM. \tag{4.17}$$

$\mathbf{Y}$ and $\mathbf{G}$ are the canonical sections of $\mathcal{V}TM$ and $\mathcal{H}TM$ respectively.

Let

$$\mathcal{H}^*TM := \mathrm{span}\Big\{ dx^1, \cdots, dx^n \Big\}, \tag{4.18}$$

$$\mathcal{V}^*TM := \mathrm{span}\Big\{ \delta y^1, \cdots, \delta y^n \Big\}, \tag{4.19}$$

where

$$\delta y^i := dy^i + N^i_j(y) dx^j. \tag{4.20}$$

We obtain a decomposition for $T^*(TM \setminus \{0\})$ which is dual to (4.17)

$$T^*(TM \setminus \{0\}) = \mathcal{V}^*TM \oplus \mathcal{H}^*TM. \tag{4.21}$$

Given a spray $\mathbf{G}$ on a manifold M. We can construct lots of sprays from $\mathbf{G}$ using the vertical radial field $\mathbf{Y}$. Let $P : TM \to \mathrm{R}$ be a function which has the following properties

(i) P is C^∞ on $TM \setminus \{0\}$,

(ii) P is positively homogeneous of degree one, i.e.,

$$P(\lambda y) = \lambda P(y), \qquad \lambda > 0.$$

Define

$$\tilde{\mathbf{G}} = \mathbf{G} - 2P\mathbf{Y}.$$

Clearly, $\tilde{\mathbf{G}}$ is a spray on M. Moreover, $\mathbf{G}$ and $\tilde{\mathbf{G}}$ have the same geodesics as point sets. That is, every geodesic of $\mathbf{G}$, after reparameterized, becomes a geodesic of $\tilde{\mathbf{G}}$, and vice versa. In this case, $\tilde{\mathbf{G}}$ is said to be *pointwise projectively related* to $\mathbf{G}$.

More general, let $\mathbf{Q} = Q^i(y)\frac{\partial}{\partial y^i}$ be a vertical vector field on $TM \setminus \{0\}$ which has the following properties

(i) $\mathbf{Q}$ is C^∞ on $TM \setminus \{0\}$,

(ii) the coefficients $Q^i(y)$ are positively homogeneous of degree two, i.e.,

$$Q^i(\lambda y) = \lambda^2 Q^i(y), \qquad \lambda > 0.$$

Define

$$\tilde{\mathbf{G}} = \mathbf{G} - 2\mathbf{Q}.$$

Again, $\tilde{\mathbf{G}}$ is a spray on M. In general, the geometric relationship between $\mathbf{G}$ and $\tilde{\mathbf{G}}$ is quite complicated.

II. Path Spaces. A spray $\mathbf{G}$ on a manifold M determines a collection of parameterized curves (called *geodesics*) in M. Conversely, a collection of parameterized curves with certain properties induces a spray. This will be explained below.

Let $\mathcal{G}$ be a collection of C^∞ parameterized curves $\sigma : (a, b) \to M$ with the following properties

(i) (Existence) For every vector $y \in T_xM$ and any t_o, there is a curve $\gamma : (a,b) \to M$ in $\mathcal{G}$ with $t_o \in (a,b)$ and $\dot\gamma(t_o) = y$;

(ii) (Uniqueness) For any two curves $\gamma(t), a < t < b$ and $\sigma(t), c < t < d$, if at some $t_0 \in (a,b)$ and $t_1 \in (c,d)$, $\dot\gamma(t_o) = \dot\sigma(t_1)$, then

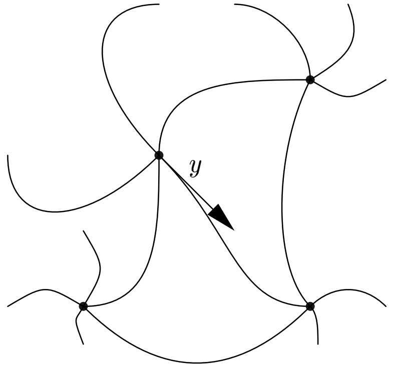
$$\gamma(t_0 + t) = \sigma(t_1 + t), \qquad t \in (a - t_o, b - t_o) \cap (c - t_1, d - t_1)$$

(iii) (Invariance) For any curve $\gamma : (a,b) \to M$ in $\mathcal{G}$ and any $t_o \in \mathrm{R}$, $\lambda > 0$ the curve

$$\tilde\gamma(t) := \gamma(\lambda t + t_o), \qquad \frac{a - t_o}{\lambda} < t < \frac{b - t_o}{\lambda}$$

is still in $\mathcal{G}$;

The pair $(M, \mathcal{G})$ is called a *path space.*



For a vector $y \in T_xM$, let $\gamma : (-\varepsilon, \varepsilon) \to M$ be the curve in $\mathcal{G}$ with $\dot\gamma(0) = y$. Define

$$G^i(y) := -\frac{1}{2}\frac{d^2\gamma^i}{dt^2}(0), \tag{4.22}$$

where $(\gamma^i(t))$ are the local coordinates of the curve $\gamma(t)$. (iii) implies

$$G^i(\lambda y) = \lambda^2 G^i(y), \qquad \lambda > 0. \tag{4.23}$$

We assert that any curve $c : (a,b) \to M$ in $\mathcal{G}$ satisfies the following system,

$$\frac{d^2c^i}{dt^2} + 2G^i\Big(\frac{dc}{dt}\Big) = 0. \tag{4.24}$$

To prove (4.24), fix a number $t \in (a,b)$ and define

$$\gamma(s) := c(s + t), \qquad a - t < s < b - t.$$

By assumption, γ belongs to $\mathcal{G}$ with

$$\frac{d\gamma}{ds}(0) = \frac{dc}{dt}(t).$$

We have

$$\frac{d^2\gamma^i}{ds^2}(0) = \frac{d^2c^i}{dt^2}(t).$$

By the definition of G^i in (4.22), we obtain

$$G^i\Big(\frac{dc}{dt}(t)\Big) = G^i\Big(\frac{d\gamma}{ds}(0)\Big) = -\frac{1}{2}\frac{d^2\gamma^i}{ds^2}(0) = -\frac{1}{2}\frac{d^2c^i}{dt^2}(t).$$

This proves (4.24). Therefore

$$\mathbf{G} := y^i\frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}.$$

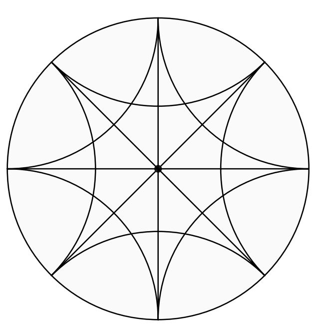
is a spray on M such that the set of geodesics of $\mathbf{G}$ is $\mathcal{G}$.

Example 4.1.4 Let B^n denote the unit ball in R^n. Take an arc C in B^n of planar circle orthogonal to the boundary $\mathrm{S}^{n-1} = \partial\mathrm{B}^n$. For every point $x \in C$, we parameterize C by $c : (a, b) \to C$ such that

(i) $c(0) = x$;

(ii) the parameter is the arc-length induced by the standard Euclidean metric;

(iii) $p = \lim_{t\to a^+} c(t)$ and $q := \lim_{t\to b^-} c(t)$ lie on S^{n-1}.



Let $\mathcal{G}$ denote the collection of the above parameterized curves in B^n. We obtain a path space $(\mathrm{B}^n, \mathcal{G})$. What is the induced spray ? ♯

Definition 4.1.3 A spray $\mathbf{G}$ is said to be *positively complete* if every geodesic defined on $[0, a)$ can be extended to a geodesic defined on $[0, \infty)$. $\mathbf{G}$ is said to be *negatively complete* if every geodesic defined on $(-a, 0]$ can be extended to a geodesic defined on $(-\infty, 0]$. A spray $\mathbf{G}$ is said to be *complete* if it is both positively and negatively complete.

There are sprays which are positively complete, but not negatively complete, hence incomplete. See Example 4.1.5 below.

Example 4.1.5 Let $\mathrm{B}^n \subset \mathrm{R}^n$ denote the Euclidean unit ball. Consider a vector field on $T\mathrm{B}^n \approx \mathrm{B}^n \times \mathrm{R}^n$,

$$\mathbf{G} := y^i \frac{\partial}{\partial x^i} - 2P(x, y) y^i \frac{\partial}{\partial y^i}, \tag{4.25}$$

where $P(x, y)$ is given by

$$P(x, y) := \frac{\langle x, y\rangle + \sqrt{|y|^2 - \Big(|x|^2|y|^2 - \langle x, y\rangle^2\Big)}}{2(1 - |x|^2)}.$$

Clearly, $\mathbf{G}$ is a spray on B^n. Geodesics of $\mathbf{G}$ are characterized by

$$\frac{d^2 x^i}{dt^2}(t) + 2P\Big(x(t), \frac{dx}{dt}(t)\Big)\frac{dx^i}{dt}(t) = 0.$$

The geodesics passing through a point $x_o \in \mathrm{B}^n$ in the direction $a \in \mathrm{S}^{n-1}$ must be straight lines in the following form

$$x(t) = f(t) a + x_o,$$

where $f(t)$ satisfies

$$f'' + \frac{f'^2}{\sqrt{1 - \delta^2 \sin^2\theta} - (f + \delta\cos\theta)} = 0 \tag{4.26}$$

with $f(0) = 0$ and $f'(0) = k > 0$, where $\delta := |x_o| < 1$ and θ denotes the angle between x_o and a. We obtain

$$f(t) = \Big(\sqrt{1 - \delta^2\sin^2\theta} - \delta\cos\theta\Big)\Big(1 - e^{\frac{-kt}{\sqrt{1-\delta^2\sin^2\theta} - \delta\cos\theta}}\Big).$$

Note that the line $\ell(s) := x_o + sa$ intersects S^{n-1} at $s = s_\pm$ where

$$s_- := -\sqrt{1 - \delta^2\sin^2\theta} - \delta\cos\theta, \qquad s_+ := \sqrt{1 - \delta^2\sin^2\theta} - \delta\cos\theta.$$

Thus the maximal interval for the solution f is $(-t_o, \infty)$ where

$$t_o := \frac{\sqrt{1 - \delta^2\sin^2\theta} - \delta\cos\theta}{k} \ln\Big(\frac{2\sqrt{1 - \delta^2\sin^2\theta}}{\sqrt{1 - \delta^2\sin^2\theta} - \delta\cos\theta}\Big).$$

This shows that $\mathbf{G}$ is only positively complete. ♯

4.2 Finsler Sprays

Let (M, L) be a Finsler space. For a curve $c : [a, b] \to M$, define

$$\mathcal{E}(c) := \int_a^b L(\dot{c}(t))dt.$$

Consider the following variational problem of L:

$$\mathcal{E}(c) = \text{Extremum}. \tag{4.27}$$

Take a variation of c, that is a map

$$H : (-\varepsilon, \varepsilon) \times [a, b] \to M$$

with fixed endpoints, i.e,

$$H(0,t) = c(t), \qquad H(u,a) = c(a), \qquad H(u,b) = c(b).$$

Let

$$\mathcal{E}(u) := \int_a^b L\Big(\frac{\partial H}{\partial t}(u,t)\Big)dt.$$

We obtain

$$\mathcal{E}'(0) = -\int_a^b g_{\dot{c}(t)}\Big(\kappa(t), V(t)\Big)dt,$$

where $V(t) := \frac{\partial H}{\partial u}(0,t)$ and $\kappa(t) = \kappa^i(t)\frac{\partial}{\partial x^i}|_{c(t)}$ is given by

$$\kappa^i(t) := \frac{d^2c^i}{dt^2}(t) + 2G^i(\dot{c}(t)), \tag{4.28}$$

where $(c^i(t))$ denotes the coordinates of $c(t)$. The coefficients $G^i(y)$ in (4.28) are given by

$$G^i(y) : \;=\; \frac{1}{4}g^{il}(y)\Big[L_{y^lx^k}(y)y^k - L_{x^l}(y)\Big] \tag{4.29}$$

$$=\; \frac{1}{4}g^{il}(y)\Big[2\frac{\partial g_{jl}}{\partial x^k}(y) - \frac{\partial g_{jk}}{\partial x^l}(y)\Big]y^jy^k. \tag{4.30}$$

Define

$$\mathbf{G} := y^i\frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}.$$

This vector field is globally defined on TM ! From the homogeneity of L, we have

$$G^i(\lambda y) = \lambda^2 G^i(y), \qquad \lambda > 0. \tag{4.31}$$

$\mathbf{G}$ is called a *Finsler spray* induced by L. The geodesics of $\mathbf{G}$ are called the *geodesics* of L. The spray coefficients G^i of $\mathbf{G}$ are also called the *spray coefficients* of L.

Let $\mathbf{G}$ be a Finsler spray induced by a Finsler metric L. Consider $\tilde{L} := \lambda L$, where $\lambda \neq 0$ is constant. We see that the spray $\tilde{\mathbf{G}}$ induced by $\tilde{L}$ is always equal to $\mathbf{G}$. In other words, if one scales the Finsler metric, the induced spray remains unchanged.

Example 4.2.1 Let L be a Finsler metric on an open subset $\mathcal{U} \subset \mathrm{R}^2$. In the standard local coordinates (x, y, u, v) in $T\mathcal{U} = \mathcal{U} \times \mathrm{R}^2$, the spray coefficients $G = G^1$ and $H = G^2$ are given by

$$G : = \frac{L_{vv}(L_{xu}u + L_{yu}v - L_x) - L_{uv}(L_{xv}u + L_{yv}v - L_y)}{2(L_{uu}L_{vv} - L_{uv}L_{uv})} \tag{4.32}$$

$$H : = \frac{-L_{uv}(L_{xu}u + L_{yu}v - L_x) + L_{uu}(L_{xv}u + L_{yv}v - L_y)}{2(L_{uu}L_{vv} - L_{uv}L_{uv})}. \tag{4.33}$$

By the homogeneity of G and H, we can simplify (4.32) and (4.33) as follows.

$$G = \frac{(L_x L_{vv} - L_y L_{uv}) - (L_{xv} - L_{yu})L_v}{2\Big(L_{uu}L_{vv} - L_{uv}L_{uv}\Big)} \tag{4.34}$$

$$H = \frac{(-L_x L_{uv} + L_y L_{uu}) + (L_{xv} - L_{yu})L_u}{2\Big(L_{uu}L_{vv} - L_{uv}L_{uv}\Big)} \tag{4.35}$$

♯

Let $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}$ be a spray induced by a Finsler metric L. Put

$$N^i_j(y) := \frac{\partial G^i}{\partial y^j}(y), \qquad \Gamma^i_{jk}(y) := \frac{\partial^2 G^i}{\partial y^j \partial y^k}(y). \tag{4.36}$$

We call N^i_j the *connection coefficients* of L and Γ^i_{jk} the *Christoffel symbols* of L. The homogeneity (4.31) implies

$$N^i_j(y) = \Gamma^i_{jk}(y)y^k, \quad 2G^i(y) = N^i_j(y)y^j = \Gamma^i_{jk}(y)y^i y^k. \tag{4.37}$$

We have the following important lemma.

Lemma 4.2.1 *For any Finsler metric on a manifold,*

$$\frac{1}{2}\Big\{\frac{\partial g_{ij}}{\partial x^k} - g_{il}\Gamma^l_{jk} - g_{jl}\Gamma^l_{ik} - 2C_{ijl}N^l_k\Big\}$$
$$= 2G^l\frac{\partial C_{ijk}}{\partial y^l} + C_{ljk}N^l_i + C_{ilk}N^l_j + C_{ijl}N^l_k - y^l\frac{\partial C_{ijk}}{\partial x^l}, \tag{4.38}$$

where C_{ijk} denote the coefficients of the Cartan torsion defined in (2.6).

Proof: Rewrite (4.30) as follows

$$g_{jl}G^l = \frac{1}{4}\Big\{2\frac{\partial g_{jk}}{\partial x^l} - \frac{\partial g_{kl}}{\partial x^j}\Big\}y^k y^l. \tag{4.39}$$

Differentiating (4.39) with respect to y^i yields

$$g_{jl}N^l_i = \frac{1}{2}\Big\{\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j}\Big\}y^k - 2C_{ijk}G^k. \tag{4.40}$$

Differentiating (4.40) with respect to y^k yields

$$\begin{aligned} g_{jl}\Gamma^l_{ik} &= \frac{1}{2}\Big\{\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j}\Big\} \\ &\quad + y^l\frac{\partial C_{ijk}}{\partial x^l} - 2G^l\frac{\partial C_{ijk}}{\partial y^l} - 2C_{ijl}N^l_k - 2C_{jkl}N^l_i. \end{aligned} \tag{4.41}$$

This implies (4.38). Q.E.D.

Lemma 4.2.2 *For any Finsler metric L on a manifold*

$$\frac{\delta}{\delta x^k}(L) = 0, \tag{4.42}$$

where $\frac{\delta}{\delta x^k} = \frac{\partial}{\partial x^k} - N^i_k(y)\frac{\partial}{\partial y^i}$. *Hence*

$$\mathbf{G}(L) = y^k\frac{\delta}{\delta x^k}(L) = 0. \tag{4.43}$$

Proof: Contracting (4.38) with $y^i y^j$ and using (2.7), we obtain

$$\frac{\partial g_{ij}}{\partial x^k}y^i y^j = 2g_{il}y^i N^l_k.$$

Thus $L = g_{ij}(y)y^i y^j$ satisfies

$$\frac{\delta}{\delta x^k}(L) = L_{x^k} - N^l_k L_{y^l} = \frac{\partial g_{ij}}{\partial x^k}y^i y^j - 2g_{il}y^i N^l_k = 0.$$

Contracting (4.42) with y^k yields (4.43). Q.E.D.

For a Finsler metric L on a manifold M, there is a canonical 1-form ω on $TM \setminus \{0\}$, defined by

$$\omega := g_{ij}(y)y^j dx^i. \tag{4.44}$$

We call ω the *Hilbert form.*

Let

$$\theta := g_{ij}(y)dx^i \wedge \delta y^j,$$

where $\delta y^i := dy^i + N^i_j(y)dx^j$. We obtain the following

Lemma 4.2.3 *The Hilbert form* ω *and the Finsler spray* $\mathbf{G}$ *are related by*

$$\omega(\mathbf{G}) = L, \qquad \theta(\mathbf{G}, \cdot) = \frac{1}{2} dL.$$

Example 4.2.2 Consider a Riemannian metric on a manifold M

$$\alpha(y) = \sqrt{a_{ij}(x) y^i y^j}.$$

Let G^k denote the spray coefficients in (4.30), which are given by

$$G^k(y) = \frac{1}{4} a^{kl}(x) \Big\{ 2 \frac{\partial a_{il}}{\partial x^j}(x) - \frac{\partial a_{ij}}{\partial x^l}(x) \Big\} y^i y^j. \tag{4.45}$$

We see that $G^k(y)$ are quadratic in $y \in T_x M$ at any point x. Let

$$\gamma^k_{ij}(x) := \frac{\partial^2 G^k}{\partial y^i \partial y^j}(x).$$

We obtain

$$\gamma^k_{ij}(x) = \frac{1}{2} a^{kl}(x) \Big\{ \frac{\partial a_{il}}{\partial x^j}(x) + \frac{\partial a_{jl}}{\partial x^i}(x) - \frac{\partial a_{ij}}{\partial x^l}(x) \Big\} \tag{4.46}$$

and

$$G^k(y) = \frac{1}{2} \gamma^k_{ij}(x) y^i y^j. \tag{4.47}$$

Now we consider the two-dimensional case. In a standard local coordinate system (x, y, u, v) in TM, express α as follows.

$$\alpha = \sqrt{a(x,y)u^2 + 2b(x,y)uv + c(x,y)v^2}.$$

Let $G := G^1$ and $H := G^2$ denote the spray coefficients of α. They are given by

$$G = \frac{1}{2}\Big(\gamma^1_{11} u^2 + 2\gamma^1_{12} uv + \gamma^1_{22} v^2 \Big) \tag{4.48}$$

$$H = \frac{1}{2}\Big(\gamma^2_{11} u^2 + 2\gamma^2_{12} uv + \gamma^2_{22} v^2 \Big), \tag{4.49}$$

where γ^k_{ij} are given by

$$\begin{aligned}
\gamma^1_{11} &= \frac{c a_x + b a_y - 2 b b_x}{2(ac - b^2)} \\
\gamma^1_{12} &= \frac{c a_y - b c_x}{2(ac - b^2)} \\
\gamma^1_{22} &= \frac{2 c b_y - c c_x - b c_y}{2(ac - b^2)} \\
\gamma^2_{11} &= \frac{2 a b_x - b a_x - a a_y}{2(ac - b^2)} \\
\gamma^2_{12} &= \frac{a c_x - b a_y}{2(ac - b^2)} \\
\gamma^2_{22} &= \frac{b c_x + a c_y - 2 b b_y}{2(ac - b^2)}.
\end{aligned}$$

♯

Example 4.2.3 Let $L = L(x, y, u, v)$ be a Finsler metric on an open subset $\mathcal{U} \subset \mathrm{R}^2$ in the form

$$L = e^{ax+by+c} N(u, v), \tag{4.50}$$

where a, b and c are constants and $N = N(u, v)$ is a Minkowski metric on $\mathcal{U}$. N is positively homogeneous of degree two, $N(\lambda u, \lambda v) = \lambda^2 N(u, v)$, $\lambda > 0$. Let $G = G^1$ and $H = G^2$ denote the spray coefficients of L. By (4.32) and (4.33), we obtain

$$\begin{aligned} G &= \frac{N_{vv}(aN_u u + bN_u v - aN) - N_{uv}(aN_v u - bN_v v + bN)}{2(N_{uu}N_{vv} - N_{uv}N_{uv})} \\ H &= \frac{-N_{uv}(aN_u u + bN_u v - aN) + N_{uu}(aN_v u - bN_v v + bN)}{2(N_{uu}N_{vv} - N_{uv}N_{uv})} \end{aligned}$$

Clearly, G and H are independent of $(x, y) \in \mathcal{U}$. By definition, L is an Antonelli metric or y-Berwald metric [An2]. ♯

4.3 Geodesics of Funk Metrics

The Funk metrics on strongly convex domains in a vector space are very important examples in Finsler geometry. We will make a close investigation on the geodesics of Funk metrics.

Let F denote the Funk metric on a strongly convex domain $\Omega \subset \mathrm{R}^n$. By Lemma 2.3.1, F satisfies

$$F_{x^k} = F F_{y^k}. \tag{4.51}$$

Using (4.29) and (4.51), we compute the spray coefficients G^i of F as follows.

$$G^i = \frac{1}{4} g^{il} \Big\{ \Big[F[F^2]_{y^k} \Big]_{y^l} y^k - F[F^2]_{y^l} \Big\} = \frac{1}{4} g^{il} F[F^2]_{y^l} = \frac{1}{2} F y^i. \tag{4.52}$$

The geodesics $c(t)$ of F are characterized by the following equations

$$\frac{d^2 c^i}{dt^2} + F(\dot{c}) \frac{dc^i}{dt} = 0. \tag{4.53}$$

Thus the geodesics of F must be straight lines in Ω.

For $x \in \Omega$ and $y \in T_x\Omega = \mathrm{R}^n$, let

$$c(s) := x + sy.$$

$c(s)$ is a geodesic of F for an appropriate parameter $s = s(t)$. Note that $c(s)$ is defined on

$$-\frac{1}{F(-y)} < s < \frac{1}{F(y)}.$$

For any s,

$$x + \frac{y}{F(y)} = x + sy + \frac{y}{F(\dot{c}(s))}.$$

We obtain

$$F(\dot{c}(s)) = \frac{F(y)}{1 - F(y)s}.$$

The F-length of c on $[0, 1/F(y))$ is

$$\int_0^{1/F(y)} F(\dot{c}(s))ds = +\infty,$$

and the F-length of c on $(-1/F(-y), 0]$ is

$$\int_{-1/F(-y)}^0 F(\dot{c}(s))ds = \ln\Big(1 + \frac{F(y)}{F(-y)}\Big) < \infty.$$

Thus F is positively complete.

Let d denote the distance induced by F. For any two points $c(s_1)$ and $c(s_2)$, where

$$-\frac{1}{F(-y)} < s_1 < s_2 < \frac{1}{F(y)},$$

the Funk distance $d(c(s_1), c(s_2))$ between $(c(s_1)$ and $c(s_2))$ is given by

$$d(c(s_1), c(s_2)) = \int_{s_1}^{s_2} \frac{F(y)}{1 - F(y)s} ds = \ln\Big(\frac{1 - F(y)s_1}{1 - F(y)s_2}\Big).$$

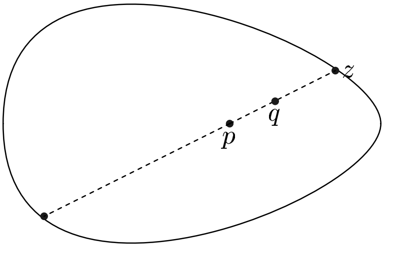
Let

$$p = c(s_1), \qquad q = c(s_2),$$

Let $|\cdot|$ denote the Euclidean norm in R^n. The above formulas for the Funk distance d can be expressed as follows

$$d(p, q) = \ln \frac{|z - p|}{|z - q|},$$

where z denotes the intersection point of the ray $\ell_{pq} := p + t(q - p)$, $t \geq 0$, with $\partial\Omega$.



For the Klein metric

$$\tilde{F}(y) := \frac{1}{2}\Big(F(-y) + F(y)\Big),$$

the spray coefficients $\tilde{G}^i$ of $\tilde{F}$ are in the form

$$\tilde{G}^i(y) = \frac{1}{2}\Big(F(y) - F(-y)\Big)\, y^i.$$

The geodesics $c(t)$ of $\tilde{F}$ are characterized by the following equations

$$\ddot{c}^i + \Big(F(\dot{c}) - F(-\dot{c})\Big)\dot{c}^i = 0. \tag{4.54}$$

Thus the geodesics of $\tilde{F}$ must be straight lines in Ω. Note that the $\tilde{F}$-length of $c(s) = x + sy$ on either side of the interval $(-1/F(-y),\ 1/F(y))$ is infinite. Thus $\tilde{F}$ is complete. From the above formulas for the Funk metric, we obtain the formula for the metric $\tilde{d}$ induced by $\tilde{F}$

$$\tilde{d}(p,q) = \frac{1}{2}\Big(d(p,q) + d(q,p)\Big).$$

See [SM] for some discussion on the asymptotic behavior of the Klein metric.

Chapter 5

S-Curvature

In this chapter, we will introduce the notion of volume form on a manifold. Meanwhile, we will define the so-called S-curvature which relates volume forms and sprays on a manifold.

5.1 Volumes

Let M be an n-dimensional manifold. A *coordinate volume form* $d\mu$ is an n-form on a coordinate neighborhood $\mathcal{U}$

$$d\mu = \sigma(x) dx^1 \cdots dx^n,$$

where $\sigma(x) > 0$. For simplicity, we will always denote by $dx^1 \cdots dx^n$ the wedge product $dx^1 \wedge \cdots \wedge dx^n$. A *volume form* on M is a collection Ξ of coordinate volume forms $\{(\mathcal{U}, d\mu)\}$ such that if $\mathcal{U} \cap \tilde{\mathcal{U}} \neq \emptyset$, then $d\mu = \sigma(x) dx^1 \cdots dx^n$ and $d\tilde{\mu} = \tilde{\sigma}(\tilde{x}) d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n$ are related by

$$\sigma = \tilde{\sigma} \Big| \frac{\partial \tilde{x}}{\partial x} \Big|.$$

We will denote a volume form on M by its representative $d\mu$ on a coordinate neighborhood $\mathcal{U}$. Every volume form $d\mu$ defines a measure μ on M by

$$\mu(f) := \int_{\mathcal{U}} f d\mu, \qquad f \in C^0(\mathcal{U}).$$

Definition 5.1.1 A *spray m space* is a triple $(M, \mathbf{G}, d\mu)$, where $\mathbf{G}$ is a spray on M and $d\mu$ is a volume form on M. A *Finsler m space* is a triple $(M, L, d\mu)$, where L is a Finsler metric on M and $d\mu$ is a volume form on M.

There are two canonical volume forms on any positive definite Finsler space. Both of them reduce to the same volume form when the Finsler metric becomes

Riemannian. In what follows, we will discuss these two canonical volume forms induced by positive definite Finsler metrics.

Let (M, F) be an n-dimensional positive definite Finsler space. Let $\{e_i\}_{i=1}^n$ be an arbitrary basis for T_xM and $\{\omega^i\}_{i=1}^n$ the dual basis for T_x^*M. The *Busemann-Hausdorff volume form* $d\mu_F$ is defined by

$$d\mu_F := \sigma_F(x)\omega^1 \wedge \cdots \wedge \omega^n, \tag{5.1}$$

where

$$\sigma_F(x) := \frac{\omega_n}{\text{Euclidean Vol}(\mathrm{B}_x^n)} \tag{5.2}$$

and

$$\mathrm{B}_x^n := \Big\{(y^i) \in \mathrm{R}^n,\ F(y^ie_i) < 1\Big\}. \tag{5.3}$$

The Busemann-Hausdorff volume form $d\mu_F$ determines a measure μ_F which is called the *Busemann-Hausdorff measure* . H. Busemann proved that if F is reversible, then μ_F is the Hausdorff measure of the induced metric d_F. See [Bu1][Bu2]. When F is Riemannian, i.e.,

$$F(y) = \sqrt{g(y,y)} = \sqrt{g_{ij}(x)y^iy^j}, \qquad y = y^ie_i \in T_xM,$$

the Busemann-Hausdorff volume form reduces to the usual Riemannian volume form. More precisely,

$$\sigma_F(x) = \sqrt{\det\big(g_{ij}(x)\big)}. \tag{5.4}$$

Example 5.1.1 Consider a Randers metric $F = \alpha + \beta$ on a manifold M with

$$\|\beta\|_x := \sup_{y\in T_xM, \alpha(y)=1} \beta(y) < 1.$$

Let $d\mu_F$ and $d\mu_\alpha$ denote the Busemann-Hausdorff volume form of F and α respectively. Let $\{e_i\}$ be an orthonormal basis for (T_xM, α_x). We may assume that $\beta_x(y) = \|\beta\|_x y^1$. Then B_x^n is a convex body in R^n given by

$$\Big(1 - \|\beta\|_x^2\Big)^2\Big(y^1 + \frac{\|\beta\|_x}{1-\|\beta\|_x^2}\Big)^2 + \Big(1 - \|\beta\|_x^2\Big)\sum_{a=2}^n (y^a)^2 < 1.$$

The Euclidean volume of B_x^n is given by

$$\mathrm{Vol}(\mathrm{B}_x^n) = \frac{\omega_n}{\big(1 - \|\beta\|_x^2\big)^{\frac{n+1}{2}}}. \tag{5.5}$$

Thus

$$d\mu_F = \Big(1 - \|\beta\|_x^2\Big)^{\frac{n+1}{2}} d\mu_\alpha. \tag{5.6}$$

This implies

$$\mu_F \le \mu_\alpha. \tag{5.7}$$

Assume that M is compact. Then

$$\mu_F(M) = \int_M \Big(1 - \|\beta\|_x^2\Big)^{\frac{n+1}{2}} d\mu_\alpha \le \mu_\alpha(M)$$

and the equality holds if and only if $\beta = 0$. ♯

Example 5.1.2 Let Ω be a strongly convex domain in R^n. The Funk metric F is defined by

$$x + \frac{y}{F_x(y)} \in \partial\Omega, \qquad y \in T_x\Omega \approx \mathrm{R}^n. \tag{5.8}$$

From the definition, the unit sphere $\mathrm{S}_x := F_x^{-1}(1)$ of F_x at x is given by

$$\mathrm{S}_x = \partial\Omega - \{x\}.$$

Take the standard orthonormal basis $\{e_i\}_{i=1}^n$ for R^n. We have

$$\mathrm{B}_x^n := \Big\{(y^i) \in \mathrm{R}^n, \quad F_x(y^i e_i) \le 1\Big\} = \bar{\Omega} - \{x\},$$

hence

$$\text{Euclidean Vol}(\mathrm{B}_x^n) = \text{Euclidean Vol}(\Omega).$$

The Busemann-Hausdorff volume form $d\mu_F = \sigma(x)dx^1 \cdots dx^n$ of F is given by

$$\sigma(x) = \frac{\omega_n}{\mathrm{Vol}(\Omega)}. \tag{5.9}$$

This implies that the Busemann-Hausdorff volume of (Ω, F) is constant

$$\mu_{\mathrm{F}}(\Omega) = \int_\Omega \sigma(x)dx^1 \cdots dx^n = \omega_n. \tag{5.10}$$

♯

There is another canonical volume form on an n-dimensional positive definite Finsler space (M, F). Let $\{e_i\}_{i=1}^n$ be an arbitrary basis for T_xM and $\{\omega^i\}_{i=1}^n$ the dual basis for T_x^*M. For a non-zero vector $y = y^i e_i \in T_xM$, put

$$g_{ij}(y) := g_y(e_i, e_j).$$

$g_{ij}(y)$ are C^∞ functions on $\mathrm{R}^n \setminus \{0\}$. Let

$$\mathrm{B}_x^n := \Big\{(y^i) \in \mathrm{R}^n, \ F(y^i e_i) < 1\Big\}. \tag{5.11}$$

Define

$$\tilde{\sigma}_F(x) := \frac{\int_{\mathrm{B}^n_x} \det\big(g_{ij}(y)\big)\, dy^1 \cdots dy^n}{\omega_n}, \tag{5.12}$$

where ω_n denotes the Euclidean volume of the standard unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$. The n-form

$$d\tilde{\mu}_F := \tilde{\sigma}_F(x)\, \omega^1 \wedge \cdots \wedge \omega^n \tag{5.13}$$

is a well-defined volume form on M. $\tilde{\mu}_F$ comes from a special volume form on $TM \setminus \{0\}$. To see this, take a look at the following natural direct sum of the cotangent space $T_y^*(TM \setminus \{0\})$ in a standard local coordinate system (x^i, y^i) in TM,

$$T_y^*(TM \setminus \{0\}) = \mathrm{span}\Big\{dx^1, \cdots, dx^n\Big\} \oplus \mathrm{span}\Big\{\delta y^1, \cdots, \delta y^n\Big\},$$

where

$$\delta y^i := dy^i + N^i_j(y) dx^j.$$

Using this decomposition, define a Riemannian metric on $TM \setminus \{0\}$ by

$$\hat{g} := g_{ij}(y) dx^i \otimes dx^j + g_{ij}(y) \delta y^i \otimes \delta y^j. \tag{5.14}$$

We call $\hat{g}$ the *Sasaki metric* induced by F. The Riemannian volume of $\hat{g}$ at $y \in TM \setminus \{0\}$ is given by

$$dV_{\hat{g}} = \det\big(g_{ij}(y)\big) dx^1 \cdots dx^n dy^1 \cdots dy^n. \tag{5.15}$$

We call $dV_{\hat{g}}$ the *Sasaki volume* induced by F. The Sasaki volume form can also be expressed in terms of the Hilbert form $\omega := g_{ij}(y) y^j dx^i$

$$(-1)^{\frac{n(n+1)}{2}} \frac{1}{n!} (d\omega)^n = \det\big(g_{ij}(y)\big) dx^1 \cdots dx^n dy^1 \cdots dy^n. \tag{5.16}$$

We should point out that P. Dazord discussed $(d\omega)^n$ in his Ph.D. thesis [Daz1]. He further defined the volume of a compact Finsler space (M, F) by

$$\mathrm{Vol}(M) := \frac{1}{\omega_n} \int_{\mathrm{B}M} dV_{\hat{g}}, \tag{5.17}$$

where $\pi : \mathrm{B}M \to M$ denote the unit ball bundle of M. Dazord actually defined the volume using the tangent sphere bundle. But his definition is essentially same as (5.17) due to the homogeneity of F. See [Daz2]. Observe that for a function f on M,

$$\int_{\mathrm{B}M} \pi^* f\, dV_{\hat{g}} = \omega_n \int_M f\, d\tilde{\mu}_F,$$

where

$$d\tilde{\mu}_F = \tilde{\sigma}_F(x) dx^1 \cdots dx^n$$

is just the volume form defined in (5.13). So with one step further from Dazord's definition, we obtain the special volume form $d\tilde{\mu}_F$ on M.

R.D. Holmes and A.C. Thompson [Tho] took a different approach to study Minkowski geometry and discovered this special volume form $d\tilde{\mu}_F$. Consequently, $d\tilde{\mu}_F$ is called the *Holmes-Thompson volume form* in literatures.

There might be many interesting volume forms associated with a Finsler metric F on a manifold M. Nevertheless, every volume form $d\mu$ on M is in the following form

$$d\mu = \phi_\mu d\mu_F,$$

where ϕ_μ is a positive C^∞ function.

Example 5.1.3 Let (V, F) be an n-dimensional Minkowski space. Fix a basis $\{e_i\}_{i=1}^n$ for V which determines a global coordinate system (x^i) in V. For any point $x \in \mathrm{V}$,

$$\tilde{\sigma}_F(x) = \tilde{\sigma}_F(0) = \frac{\int_{\mathrm{B}^n} \det\big(g_{ij}(y)\big)\ dy^1 \cdots dy^n}{\omega_n},$$

where $g_{ij}(y) = g_y(e_i, e_j)$ and

$$\mathrm{B}^n := \Big\{(y^i) \in \mathrm{R}^n,\ F(y^i e_i) < 1\Big\}.$$

The volume of the unit ball B of F with respect to $d\tilde{\mu}_F$ is given by

$$\tilde{\mu}_F(\mathrm{B}) = \frac{\text{Euclidean Vol}(\mathrm{B}^n)}{\omega_n} \int_{\mathrm{B}^n} \det\big(g_{ij}(y)\big) dy^1 \cdots dy^n.$$

Assume that F is reversible, i.e., $F(-y) = F(y)$. According to [Du], the Santaló's inequality [MePa] implies

$$\text{Euclidean Vol}(\mathrm{B}^n) \int_{\mathrm{B}^n} \det\big(g_{ij}(y)\big) dy^1 \cdots dy^n \le (\omega_n)^2$$

and the equality holds if and only if F is Euclidean. Thus

$$\tilde{\mu}_F(\mathrm{B}) \le \omega_n = \mu_F(\mathrm{B})$$

and the equality holds if and only if F is Euclidean. This implies that for any reversible positive definite Finsler metric F on a manifold M,

$$\tilde{\mu}_F \le \mu_F \tag{5.18}$$

and the equality holds if and only if F is Riemannian. $\sharp$

5.2 S-Curvature

Consider the trivial spray on R^n

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i}$$

and a family of volume forms $d\mu_\varepsilon = \sigma_\varepsilon(x) dx^1 \cdots dx^n$ on R^n, where

$$\sigma_\varepsilon(x) = \varepsilon^{\frac{n}{2}} e^{-\varepsilon|x|^2}, \qquad \varepsilon > 0. \tag{5.19}$$

$d\mu_\varepsilon$ determines the well-known Gaussian measures μ_ε on R^n. Note that $\sigma_\varepsilon(x) \to 0$ as $\varepsilon \to 0^+$. For a fixed value of ε, the *decay rate* of the Gaussian measure μ_ε at x is defined by

$$\mathbf{S}_x(y) := -\frac{y^i}{\sigma_\varepsilon(x)} \frac{\partial \sigma_\varepsilon}{\partial x^i}(x) = 2\varepsilon\langle x, y\rangle, \qquad y \in T_x R^n = R^n.$$

The decay rate $\mathbf{S}_x$ of μ_ε approaches ∞ (in the radial direction) as $x \to \infty$. We can extend the notion of decay rate to spray m spaces.

Let $(M, \mathbf{G}, d\mu)$ be a spray m space. In a standard local coordinate system (x^i, y^i) in TM, put

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(y) \frac{\partial}{\partial y^i}, \qquad d\mu = \sigma(x) dx^1 \cdots dx^n.$$

Define

$$\mathbf{S}(y) := \frac{\partial G^m}{\partial y^m}(y) - \frac{y^m}{\sigma(x)} \frac{\partial \sigma}{\partial x^m}(x), \qquad y = y^i \frac{\partial}{\partial x^i}|_x \in T_x M. \tag{5.20}$$

$\mathbf{S}$ has the following homogeneity property.

$$\mathbf{S}(\lambda y) = \lambda \mathbf{S}(y), \qquad \lambda > 0, \ y \in TM \setminus \{0\}. \tag{5.21}$$

The quantity $\mathbf{S}$ is originally defined by the author [Sh3] for Finsler spaces with the induced Busemann-Hausdorff measure. $\mathbf{S}$ is called the *mean covariation* in [Sh3] and the *mean tangent curvature* in [Sh5]. The mean tangent curvature is generalized to spray m spaces $(M, \mathbf{G}, d\mu)$ for the first time in this book. We shall call $\mathbf{S}$ the *S-curvature*.

The following four types of sprays m spaces are of interest:

(S1) $\mathbf{S}(y) = S_i(x) y^i$ is a 1-form in y;

(S2) $\mathbf{S}(y) = S_i(x) y^i$ is a close 1-form on M;

(S3) $\mathbf{S}(y) = S_i(x)y^i$ is an exact form on M;

(S4) $\mathbf{S} = 0$.

By (5.20), we see that (S1) (S2) (S3) are independent of the underlying volume form $d\mu$. Clearly, if $G^i(y)$ are quadratic in $y \in T_xM$, $x \in M$, then $\mathbf{S}$ becomes a 1-form on M.

Every spray on a regular measure space can be deformed to another spray with vanishing S-curvature. More precisely, we have

Proposition 5.2.1 *Let $(M, \mathbf{G}, d\mu)$ be a spray m space and*

$$\tilde{\mathbf{G}} := \mathbf{G} + \frac{2\mathbf{S}}{n+1}\mathbf{Y}, \tag{5.22}$$

where $\mathbf{Y} = y^i \frac{\partial}{\partial y^i}$ denotes the canonical vertical vector field on TM. The S-curvature of $(M, \tilde{\mathbf{G}}, d\mu)$ always vanishes.

Proof: In local coordinates,

$$\tilde{G}^i(y) = G^i(y) - \frac{\mathbf{S}(y)}{n+1}y^i. \tag{5.23}$$

By (5.23), we obtain

$$\frac{\partial \tilde{G}^i}{\partial y^j} = \frac{\partial G^i}{\partial y^j} - \frac{1}{n+1}\frac{\partial \mathbf{S}}{\partial y^j}y^i - \frac{\mathbf{S}}{n+1}\delta^i_j. \tag{5.24}$$

The homogeneity of $\mathbf{S}$ implies

$$\frac{\partial \mathbf{S}}{\partial y^i}y^i = \mathbf{S}.$$

Thus

$$\frac{\partial \tilde{G}^m}{\partial y^m} = \frac{\partial G^m}{\partial y^m} - \mathbf{S}. \tag{5.25}$$

We obtain

$$\tilde{\mathbf{S}} = \frac{\partial \tilde{G}^m}{\partial y^m} - \frac{y^m}{\sigma}\frac{\partial \sigma}{\partial x^m} = \frac{\partial G^m}{\partial y^m} - \frac{y^m}{\sigma}\frac{\partial \sigma}{\partial x^m} - \mathbf{S} = 0.$$

With this, the proof is completed. Q.E.D.

Example 5.2.1 Let F denote the Funk metric on a strongly convex domain Ω in R^n. According to Section 4.3, the spray coefficients G^i of F are given by

$$G^i = \frac{1}{2}F\, y^i. \tag{5.26}$$

Differentiating (5.26) yields

$$\frac{\partial G^m}{\partial y^m} = \frac{n+1}{2} F.$$

By Example 5.1.2, the Busemann-Hausdorff volume form $d\mu_F = \sigma_F(x) dx^i \cdots dx^n$ is given by

$$\sigma_F = \frac{\omega_n}{\text{Euclidean Vol}(\Omega)} = constant.$$

Plugging them into (5.20), one obtains

$$\mathbf{S}(y) = \frac{n+1}{2} F(y). \tag{5.27}$$

In this sense, the S-curvature of F is constant. ♯

Example 5.2.2 Consider a Randers metric $F = \alpha + \beta$ on a manifold M, where $\alpha(y) = \sqrt{a_{ij} y^i y^j}$ is a Riemannian metric and $\beta(y) = b_i y^i$ is a 1-form with

$$\|\beta\|_\alpha(x) := \sup_{y \in T_x M} \frac{\beta(y)}{\alpha(y)} = \sqrt{a^{ij} b_i b_j} < 1.$$

Define $b_{i;j}$ by

$$b_{i;j} := \frac{\partial b_i}{\partial x^j} - b_k \gamma^k_{ij},$$

where γ^i_{jk} denote the Christoffel symbols of α. Let

$$r_{ij} := \frac{1}{2}(b_{i;j} + b_{j;i}), \qquad s_{ij} := \frac{1}{2}(b_{i;j} - b_{j;i}).$$

Let G^i and $\bar{G}^i$ denote the spray coefficients of F and α respectively (see (4.30)). They are related by

$$G^i = \bar{G}^i + P y^i + Q^i, \tag{5.28}$$

where

$$P : = \frac{1}{2F}\Big(r_{kl} y^k y^l - 2\alpha(y) b_r a^{rp} s_{pl} y^l \Big) \tag{5.29}$$

$$Q^i : = \alpha a^{ir} s_{rl} y^l. \tag{5.30}$$

See [BaChSh1] for a proof.

Differentiating (5.28) with respect to y^i, then taking summation over i, we obtain

$$N^m_m = \bar{N}^m_m + (n+1)P. \tag{5.31}$$

Let

$$d\mu_F = \sigma_F(x) dx^1 \cdots dx^n, \qquad d\mu_\alpha = \sigma_\alpha(x) dx^1 \cdots dx^n.$$

From Example 5.1.1,

$$\sigma_F(x) = \Big(1 - \|\beta\|^2_\alpha(x)\Big)^{\frac{n+1}{2}} \sigma_\alpha(x), \tag{5.32}$$

Note

$$\frac{y^k}{\sigma_\alpha}\frac{\partial \sigma_\alpha}{\partial x^k} = \bar{N}^m_m. \tag{5.33}$$

Plugging (5.31) and (5.32) into (5.20) yields

$$\mathbf{S}(y) = (n+1)\Big[\frac{1}{2(1-\|\beta\|^2_\alpha(x))}(\|\beta\|^2_\alpha)_{x^j} y^j + P(y)\Big]. \tag{5.34}$$

By an elementary argument, one can easily prove that the following conditions are equivalent for Randers metrics.

(i) $\mathbf{S} = 0$;

(ii) $\mathbf{S}$ is a 1-form;

(iii) β satisfies

$$r_{ij} = -\frac{1}{2(1-\|\beta\|^2_\alpha)}\Big[b_i(\|\beta\|^2_\alpha)_{x^j} + b_j(\|\beta\|^2_\alpha)_{x^i}\Big]. \tag{5.35}$$

Thus if

$$r_{ij} = 0, \qquad \|\beta\|_\alpha = constant, \tag{5.36}$$

then $\mathbf{S} = 0$. ♯

The condition (5.36) means that β is a Killing 1-form of constant length. If β is parallel, then it must be a Killing 1-form of constant length. There are some Riemannian spaces with non-parallel Killing 1-forms of constant length. See Example 5.2.3 below.

Example 5.2.3 Let $\{\zeta^1, \zeta^2, \zeta^3\}$ be the canonical left-invariant co-frame on the Lie group $\mathrm{Sp}(1) = \mathrm{S}^3$ so that the following is the canonical Riemannian metric of constant curvature $\kappa = 1$ on S^3.

$$\alpha_1(y) := \sqrt{[\zeta^1(y)]^2 + [\zeta^2(y)]^2 + [\zeta^3(y)]^2}$$

$\zeta^1, \zeta^2, \zeta^3$ satisfy

$$d\zeta^1 = 2\zeta^2 \wedge \zeta^3, \qquad d\zeta^2 = 2\zeta^3 \wedge \zeta^1, \qquad d\zeta^3 = 2\zeta^1 \wedge \zeta^2. \tag{5.37}$$

Consider $F = \alpha + \beta$, where

$$\alpha(y) : \;=\; \sqrt{a^2[\zeta^1(y)]^2 + b^2[\zeta^2(y)]^2 + c^2[\zeta^3(y)]^2}, \tag{5.38}$$

$$\beta(y) : \;=\; b_1\zeta^1(y) + b_2\zeta^2(y) + b_3\zeta^3(y). \tag{5.39}$$

Assume that

$$\|\beta\|_\alpha = \sqrt{\Big(\frac{b_1}{a}\Big)^2 + \Big(\frac{b_2}{b}\Big)^2 + \Big(\frac{b_3}{c}\Big)^2} < 1.$$

Then F is a special Randers metric. Let

$$\omega^1 := a\zeta^1, \qquad \omega^2 := b\zeta^2, \qquad \omega^3 := c\zeta^3.$$

$\{\omega^1, \omega^2, \omega^3\}$ is an orthonormal co-frame for $T\mathrm{S}^3$ with respect to α. It follows from (5.37) that

$$d\omega^i = \omega^j \wedge \omega_j{}^i,$$

where $\omega_j{}^i$ are the Levi-Civita connection forms of α, determined by $\omega_j{}^i + \omega_i{}^j = 0$,

$$\omega\ {}^1 = cA\ \zeta^3, \qquad \omega_3{}^1 = -bB\ \zeta^2, \qquad \omega_3{}^2 = aC\ \zeta^1,$$

and

$$\begin{aligned} A &= \frac{a}{bc} + \frac{b}{ac} - \frac{c}{ab} \\ B &= \frac{a}{bc} + \frac{c}{ab} - \frac{b}{ac} \\ C &= \frac{b}{ac} + \frac{c}{ab} - \frac{a}{bc}. \end{aligned}$$

Express β in the form

$$\beta = \tilde{b}_1\omega^1 + \tilde{b}_2\omega^2 + \tilde{b}_3\omega^3$$

where $\tilde{b}_1 = b_1/a, \tilde{b}_2 = b_2/b$ and $\tilde{b}_3 = b_3/c$. Define $\nabla\beta = \tilde{b}_{ij}\omega^i \otimes \omega^j$. A direct computation gives

$$\tilde{b}_{11} = 0, \qquad \tilde{b}_{12} = -\tilde{b}_3 B, \qquad \tilde{b}_{13} = \tilde{b}_2 A$$

$$\tilde{b}_{21} = \tilde{b}_3 C, \qquad \tilde{b}_{22} = 0, \qquad \tilde{b}_{23} = -\tilde{b}_1 A$$

$$\tilde{b}_{31} = -\tilde{b}_2 C, \qquad \tilde{b}_{32} = \tilde{b}_1 B, \qquad \tilde{b}_{33} = 0.$$

Let $\{e_1, e_2, e_3\}$ be the frame dual to $\{\omega^1, \omega^2, \omega^3\}$. For $y = ue_1 + ve_2 + we_3$,

$$\begin{aligned} P &= \frac{1}{F(y)abc}\Big\{ \frac{b_3}{c}\Big(b^2 - a^2\Big)uv + \frac{b_2}{b}\Big(a^2 - c^2\Big)uw + \frac{b_1}{a}\Big(c^2 - b^2\Big)vw \\ &\quad -\alpha\frac{b_2 b_3}{bc}\Big(c^2 - b^2\Big)u - \alpha\frac{b_1 b_3}{ac}\Big(a^2 - c^2\Big)v - \alpha\frac{b_1 b_2}{ab}\Big(b^2 - a^2\Big)w\Big\}. \end{aligned}$$

Since $\|\beta\|_\alpha = constant$, (5.34) simplifies to

$$\mathbf{S}(y) = (n+1)P.$$

Thus $\mathbf{S} = 0$ if and only if

$$b_1 b_2\Big(a^2 - b^2\Big) = 0, \qquad b_3\Big(a^2 - b^2\Big) = 0,$$

$$b_1 b_3 \left(a^2 - c^2\right) = 0, \quad b_2 \left(a^2 - c^2\right) = 0,$$
$$b_2 b_3 \left(b^2 - c^2\right) = 0, \quad b_1 \left(b^2 - c^2\right) = 0.$$

As one can see, there are some non-trivial solutions for the above system of algebraic equations. ♯

The S-curvature can be expressed in an index-free form using geodesic fields. The index-free form is particularly useful in comparison theorem.

Let $(M, L, d\mu)$ be a Finsler m space. Let $\{e_i\}_{i=1}^n$ be an arbitrary basis for $T_x M$ and $\{\omega^i\}_{i=1}^n$ the basis for $T_x^* M$ dual to $\{e_i\}_{i=1}^n$. Put

$$d\mu := \sigma(x) \omega^1 \wedge \cdots \wedge \omega^n.$$

Assume that the index of L, $\mathrm{ind}(L) = 0$. Thus $\det[g_y(e_i, e_j)] > 0$. Define

$$\tau(y) := \ln \left[\frac{\sqrt{\det(g_y(e_i, e_j))}}{\sigma(x)} \right], \quad y \in T_x M \setminus \{0\}. \tag{5.40}$$

τ is a positively homogeneous function of $y \in T_x M$ of degree zero. τ is called the *distortion* of $(L, d\mu)$. Note that $\tau = 0$ when L is Riemannian and $d\mu$ is the Riemannian volume form.

A non-zero vector field Y on an open subset $\mathcal{U}$ in M is said to be *geodesic* if the integral curves of Y are geodesics, namely,

$$Y^j \frac{\partial Y^i}{\partial x^j} + 2G^i(Y) = 0. \tag{5.41}$$

Lemma 5.2.2 ([Sh3][Sh5]) *Let (M, L) be a Finsler space with $\det(g_{ij}) > 0$. For a tangent vector $y \in T_x M$, let Y be a geodesic field with $Y_x = y$. Then*

$$\mathbf{S}(y) = y\Big[\tau(Y)\Big]. \tag{5.42}$$

Proof: Observe that in local coordinates

$$\frac{\partial}{\partial x^k}\Big(\ln \sqrt{\det(g_{ij})}\Big) = \frac{1}{2} g^{ij} \frac{\partial g_{ij}}{\partial x^k}, \tag{5.43}$$
$$\frac{\partial}{\partial y^k}\Big(\ln \sqrt{\det(g_{ij})}\Big) = g^{ij} C_{ijk}, \tag{5.44}$$

where C_{ijk} denote the coefficients of the Cartan torsion. It follows from (4.40) that

$$\frac{1}{2} y^k g^{ij} \frac{\partial g_{ij}}{\partial x^k} - 2 g^{ij} C_{ijl} G^l = N_i^i. \tag{5.45}$$

By definition, the geodesic field $Y = Y^i \frac{\partial}{\partial x^i}$ satisfies

$$Y^k \frac{\partial Y^i}{\partial x^k} + 2G^i(Y) = 0. \tag{5.46}$$

Here Y in G^i stands for its coordinates $(x^1, \cdots, x^n; Y^1(x), \cdots, Y^n(x))$. By (5.43)-(5.46), we obtain

$$\begin{aligned} Y\Big[\tau(Y)\Big] &= Y^k \frac{\partial}{\partial x^k}\Big(\ln \sqrt{\det(g_{ij}(Y))}\Big) - Y^k \frac{\partial}{\partial x^k}\Big(\ln \sigma\Big) \\ &= \frac{1}{2} Y^k g^{ij}(Y)\Big[\frac{\partial g_{ij}}{\partial x^k}(Y) + 2C_{ijl}(Y)\frac{\partial Y^l}{\partial x^k}\Big] - \frac{Y^i}{\sigma}\frac{\partial \sigma}{\partial x^i} \\ &= \frac{1}{2} Y^k g^{ij}(Y)\frac{\partial g_{ij}}{\partial x^k}(Y) - 2g^{ij}(Y)C_{ijl}(Y)G^l(Y) - \frac{Y^i}{\sigma}\frac{\partial \sigma}{\partial x^i} \\ &= N^i_i(Y) - \frac{Y^k}{\sigma}\frac{\partial \sigma}{\partial x^k} = \mathbf{S}(Y). \end{aligned}$$

This completes the proof. Q.E.D.

5.3 Geodesic Flows

In this section, we will discuss geodesic flows and the geometric meaning of the S-curvature.

Let $\mathbf{G}$ be a spray on M. For any $y \in TM \setminus \{0\}$, there is an unique curve $\hat{c}(t) = \hat{c}_y(t)$ in $TM \setminus \{0\}$ satisfying

$$\frac{d\hat{c}}{dt} = \mathbf{G}_{\hat{c}}, \qquad \hat{c}(0) = y. \tag{5.47}$$

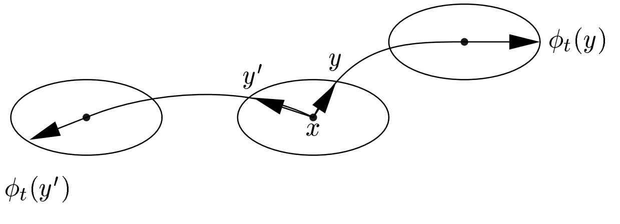
Let $\phi_t(y) := \hat{c}_y(t)$. We obtain a family of diffeomorphisms

$$\phi_t : TM \setminus \{0\} \to TM \setminus \{0\}.$$

Each ϕ_t has the following homogeneity property

$$\phi_t(\lambda y) = \lambda \phi_t(y), \qquad \lambda > 0.$$

We call ϕ_t the *geodesic flow* of $\mathbf{G}$.



Let $d\mu = \sigma(x)dx^1 \cdots dx^n$ be a volume form on an n-dimensional manifold M. $d\mu$ induces a volume form $d\hat{\mu}$ on TM by

$$d\hat{\mu} := \sigma^2(x)dx^1 \cdots dx^n dy^1 \cdots dy^n. \tag{5.48}$$

We have the following

Proposition 5.3.1 *Let $(M, \mathbf{G}, d\mu)$ be a spray m space and ϕ_t denote the geodesic flow of $\mathbf{G}$. Then*

$$\frac{d}{dt}\Big((\phi_t)^* d\hat{\mu}\Big)|_{t=0} = -2\mathbf{S}\ d\hat{\mu}. \tag{5.49}$$

Hence if $\mathbf{S} = 0$, then the geodesic flow ϕ_t preserves $d\hat{\mu}$.

Proof: In the standard local coordinates (x^i, y^i), let (φ^i_t, ψ^i_t) denote the coordinates of ϕ_t for small t. Then $\varphi^i_0(y) = x^i$ and $\psi^i_0(y) = y^i$. For every $y \in TM\setminus\{0\}$, $\hat{c}(t) := \phi_t(y)$ is an integral curve of $\mathbf{G}$, i.e.,

$$\frac{d\varphi^i_t}{dt} = \psi^i_t, \tag{5.50}$$

$$\frac{d\psi^i_t}{dt} = -2G^i\Big(\frac{d\phi_t}{dt}(y)\Big). \tag{5.51}$$

Observe that

$$(\phi_t)^* dx^i = \frac{\partial \varphi^i_t}{\partial x^k} dx^k + \frac{\partial \varphi^i_t}{\partial y^k} dy^k,$$

$$(\phi_t)^* dy^i = \frac{\partial \psi^i_t}{\partial x^k} dx^k + \frac{\partial \psi^i_t}{\partial y^k} dy^k.$$

By (5.50) and (5.51), we obtain

$$\frac{d}{dt}\Big((\phi_t)^* dx^i\Big)|_{t=0} = dy^i, \tag{5.52}$$

$$\frac{d}{dt}\Big((\phi_t)^* dy^i\Big)|_{t=0} = -2\frac{\partial G^i}{\partial x^k}(y) dx^k - 2\frac{\partial G^i}{\partial y^k}(y) dy^k. \tag{5.53}$$

It follows from (5.52) and (5.53) that

$$\frac{d}{dt}\Big((\phi_t)^*\Big[dx^1 \cdots dx^n dy^1 \cdots dy^n\Big]\Big)|_{t=0} = -2\frac{\partial G^m}{\partial y^m} dx^1 \cdots dx^n dy^1 \cdots dy^n. \tag{5.54}$$

On the other hand,

$$\frac{d}{dt}\Big(\sigma^2(\pi \circ \phi_t)\Big)|_{t=0} = 2\sigma(x) y^m \frac{\partial \sigma}{\partial x^m}(x). \tag{5.55}$$

Then

$$\begin{aligned}\frac{d}{dt}\Big((\phi_t)^* d\hat{\mu}\Big)|_{t=0} &= -2\sigma^2(x)\Big[\frac{\partial G^m}{\partial y^m}(y) - \frac{y^m}{\sigma(x)}\frac{\partial \sigma}{\partial x^m}(x)\Big] dx^1 \cdots dx^n dy^1 \cdots dy^n \\ &= -2\mathbf{S}(y) d\hat{\mu}.\end{aligned}$$

This gives (5.49). Q.E.D.

Given a spray m space $(M, \mathbf{G}, d\mu)$. Consider another spray

$$\tilde{\mathbf{G}} := \mathbf{G} + \frac{2\mathbf{S}}{n+1}\mathbf{Y}.$$

By Proposition 5.2.1, we know that the S-curvature $\tilde{\mathbf{S}}$ of $\tilde{\mathbf{G}}$ vanishes. Thus the geodesic flow $\tilde{\phi}_t$ of $\tilde{\mathbf{G}}$ preserves the volume form $d\hat{\mu}$ on TM induced by $d\mu$.

Corollary 5.3.2 *Let* $(M, \mathbf{G}, d\mu)$ *be an arbitrary spray m space and* $\tilde{\phi}_t$ *denote the geodesic flow of* $\tilde{\mathbf{G}} = \mathbf{G} + \frac{2\mathbf{S}}{n+1}\mathbf{Y}$. *Then* $\tilde{\phi}_t$ *preserves the volume form* $d\hat{\mu}$ *on* TM *induced by* $d\mu$.

Let (M, F) be a positive definite Finsler space. F induces the Sasaki metric $\hat{g}$, and $\hat{g}$ induces the Sasaki volume form $dV_{\hat{g}}$ on $TM \setminus \{0\}$. See (5.14) and (5.15). In his Ph.D. thesis [Daz1][Daz2], P. Dazord proved that the geodesic flow $\phi_t : TM \setminus \{0\} \to TM \setminus \{0\}$ always preserves the Sasaki volume form $dV_{\hat{g}}$. Thus, the Sasaki metric and the Sasaki volume form on $TM \setminus \{0\}$ play an important role in Finsler geometry.

Chapter 6

Non-Riemannian Quantities

In this chapter, we will introduce several geometric quantities for sprays and Finsler metrics. These quantities always vanish for Riemannian spaces. Therefore, they are said to be non-Riemannian. The most important non-Riemannian quantity was introduced by L. Berwald [Bw1][Bw2][Bw8] for Finsler metrics. Who is L. Berwald ?

Ludwig Berwald

Lugwig Berwald was born on December 8, 1883, in Prague. He always listed his religion in the school catalog as Jewish. He entered the Royal Ludwig-Maximilian University (Munich) in the Fall of 1892, where he pursued his studies in mathematics and physics. In December of 1908, he earned his doctorate under professor Auriel Voss with his paper "Über die Krümmungseigenschaften der Brennflachen eines geradlinigen Strahlsystems und der in ihm enthaltenen Regelflächen". In 1915, Berwald married Hedwig Adler (born in Prague). After then, he became a lecturer at the German University in Prague. He was promoted to an associate professorship in 1922, and became a full professor in 1924.

On October 22, 1941, Berwald's scientific work came to a close. The Berwalds were deported to the Ghetto in Lodz, Poland, by order of the German Secret Police. Mrs Berwald died on March 27, 1942. Cause of death: Blocked arteries. Professor Ludwig Berwald died a few weeks later on April 20th. Cause of death: Intestinal catarrh, heart failure.

L. Berwald's scientific work is mainly in the area of differential geometry. He wrote 54 papers until the time of his deportation. Only a small fraction of his investigations [Bw1]-[Bw8] are mentioned in this book.

6.1 Berwald Curvature of Sprays

Let $(M, \mathbf{G})$ be a spray space. In a standard local coordinate system (x^i, y^i) in TM, $\mathbf{G}$ is expressed in the following form

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(y) \frac{\partial}{\partial y^i},$$

where G^i satisfy

$$G^i(\lambda y) = \lambda^2 G^i(y), \qquad \lambda > 0.$$

Let

$$N^i_j(y) := \frac{\partial G^i}{\partial y^j}(y), \qquad \Gamma^i_{jk}(y) := \frac{\partial^2 G^i}{\partial y^j \partial y^k}(y). \tag{6.1}$$

N^i_j and Γ^i_{jk} are called the connection coefficients and the Christoffel symbols of **G** respectively. Note that Γ^i_{jk} are local functions on $TM\setminus\{0\}$. When we change the local coordinate system (x^i) in the base manifold M, the corresponding standard local coordinate system (x^i, y^i) in TM changes and Γ^i_{jk} change subject to the transformation law. More precisely, if the local coordinate system (x^i) is changed to another local coordinate system $(\tilde{x}^i)$, then

$$\tilde{\Gamma}^p_{qr} = \frac{\partial \tilde{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \tilde{x}^q} \frac{\partial x^k}{\partial \tilde{x}^r} \Gamma^i_{jk} + \frac{\partial \tilde{x}^p}{\partial x^i} \frac{\partial^2 x^i}{\partial \tilde{x}^q \partial \tilde{x}^r}. \tag{6.2}$$

Note that Γ^i_{jk} depend on (y^i) and $\tilde{\Gamma}^p_{qr}$ depend on $(\tilde{y}^p)$, where $\tilde{y}^p = \frac{\partial \tilde{x}^p}{\partial x^i} y^i$. But the second term on the right hand side of (6.2) is independent of (y^i).

Differentiating (6.2) with respect to y^l yields

$$\frac{\partial \tilde{\Gamma}^p_{qr}}{\partial \tilde{y}^s} \frac{\partial \tilde{x}^s}{\partial x^l} = \frac{\partial \tilde{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \tilde{x}^q} \frac{\partial x^k}{\partial \tilde{x}^r} \frac{\partial \Gamma^i_{jk}}{\partial y^l}. \tag{6.3}$$

From (6.3), we see that $\dfrac{\partial \Gamma^i_{jk}}{\partial y^l}$ are the coefficients of a tensor on $TM \setminus \{0\}$. Set

$$B^i_{jkl}(y) := \frac{\partial \Gamma^i_{jk}}{\partial y^l}(y) = \frac{\partial^2 N^i_j}{\partial y^k \partial y^l}(y) = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}(y). \tag{6.4}$$

This leads to an important quantity. For a tangent vector $y \in T_xM\setminus\{0\}$, define

$$\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$$

by

$$\mathbf{B}_y(u, v, w) := B^i_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i}|_x, \tag{6.5}$$

$u = u^i \frac{\partial}{\partial x^i}|_x, v = v^j \frac{\partial}{\partial x^j}|_x$ and $w = w^k \frac{\partial}{\partial x^k}|_x$. $\mathbf{B}_y(u, v, w)$ is symmetric in u, v and w. The homogeneity of G^i implies

$$\mathbf{B}_y(y, v, w) = 0. \tag{6.6}$$

L. Berwald first noticed that the third order derivatives of G^i give rise to an invariant for Finsler metrics [Bw1][Bw2]. J. Douglas extended the notion to sprays without any modification [Dg1]. Therefore, we make the following

Definition 6.1.1 **B** is called the *Berwald curvature*A spray is said to be *affine* if $\mathbf{B} = 0$. .

There is another interesting quantity for sprays. For $u, v \in T_xM \setminus \{0\}$, let

$$N^i(u,v) := v^j N^i_j(u) - u^j N^i_j(v).$$

Clearly, $N^i(u,v)$ can be extended to $u = 0$ or $v = 0$. Define $\mathbf{N} : T_xM \times T_xM \to T_xM$ by

$$\mathbf{N}(u,v) = N^i(u,v)\frac{\partial}{\partial x^i}|_x. \tag{6.7}$$

$\mathbf{N}(u,v)$ is well-defined. From the definition,

$$\mathbf{N}(u,v) + \mathbf{N}(v,u) = 0.$$

We call $\mathbf{N}$ the *N-curvature*. By an easy argument, we can show that $\mathbf{B} = 0$ if and only if $\mathbf{N} = 0$. Thus the following are equivalent.

(i) $\mathbf{B} = 0$;

(ii) $\mathbf{N} = 0$;

(iii) the spray coefficients $G^i(y)$ are quadratic in $y \in T_xM$ for all $x \in M$;

(iv) $\Gamma^i_{jk}(y)$ are independent of $y \in T_xM \setminus \{0\}$ for all $x \in M$.

For an affine spray $\mathbf{G}$, we obtain a set of local functions $\Gamma^i_{jk}(x) := \frac{\partial G^i}{\partial y^j \partial y^k}$ on M. Using them, we can define

$$\mathrm{D} : T_xM \times C^\infty(TM) \to T_xM$$

by

$$\mathrm{D}_y U := \Big\{ dU^i(y) + U^j(x)\Gamma^i_{jk}(x)y^k \Big\} \frac{\partial}{\partial x^i}|_x. \tag{6.8}$$

The family of maps D has the following properties

(a) $\mathrm{D}_y(fU) = y(f)U + f\mathrm{D}_yU$;

(b) $\mathrm{D}_y(U+V) = \mathrm{D}_yU + \mathrm{D}_yV$;

(c) $\mathrm{D}_{\lambda y}U = \lambda \mathrm{D}_yU$;

(d) $\mathrm{D}_{y+v}U = \mathrm{D}_yU + \mathrm{D}_vU$;

(e) $\mathrm{D}_UV - \mathrm{D}_VU = [U,V]$;

where $\lambda \in \mathrm{R}, f \in C^\infty(M), y, v \in T_xM, U, V \in C^\infty(TM)$. D is smooth in the sense that $\mathrm{D}_UV \in C^\infty(TM)$. A family of maps D with the above properties is called an *affine connection* on TM. Therefore, affine sprays on a manifold are nothing but affine connections on the tangent bundle. H. Weyl first introduced the notion of affine connections as a generalization of the Levi-Civita connection of Riemannian metrics from his continuum-based view of differential geometry.

Weyl claimed over and over again that Riemannian geometry did not appear completely convincing [We2]. This probably leads to the study of general sprays.

Let

$$E_{jk}(y) := \frac{1}{2} B^m_{jkm}(y), \tag{6.9}$$

This set of local functions give rise to a tensor on $TM \setminus \{0\}$. For a tangent vector $y \in T_xM \setminus \{0\}$, define

$$\mathbf{E}_y : T_xM \times T_xM \to \mathrm{R}$$

by

$$\mathbf{E}_y(u,v) := E_{jk}(y) u^j v^k, \tag{6.10}$$

where $u = u^i \frac{\partial}{\partial x^i}|_x$ and $v = v^j \frac{\partial}{\partial x^j}|_x$. $\mathbf{E}_y(u,v)$ is symmetric in u and v. $\mathbf{E}_y$ can be viewed as the trace of $\mathbf{B}_y$. (6.6) implies

$$\mathbf{E}_y(y,v) = 0. \tag{6.11}$$

Definition 6.1.2 **E** is called the *mean Berwald curvature*. A spray is called a *weakly affine spray* if $\mathbf{E} = 0$.

From (6.4), we have

$$E_{jk}(y) = \frac{1}{2} \frac{\partial \Gamma^m_{jk}}{\partial y^m}(y) = \frac{1}{2} \frac{\partial^2 N^m_m}{\partial y^j \partial y^k}(y) = \frac{1}{2} \frac{\partial^3 G^m}{\partial y^j \partial y^k \partial y^m}(y). \tag{6.12}$$

Suppose that M is equipped with a volume form $d\mu$. Then the S-curvature is defined for the pair $(\mathbf{G}, d\mu)$. It follows from (5.20) and (6.12) that

$$E_{ij}(y) = \frac{1}{2} \mathbf{S}_{y^i y^j}(y). \tag{6.13}$$

From (6.13), we immediately obtain the following

Proposition 6.1.3 *For a spray m space* $(M, \mathbf{G}, d\mu)$, $\mathbf{E} = 0$ *if and only if* $\mathbf{S}$ *is a* 1*-form on* M*.*

From (6.9), we have

$$\mathbf{B} = 0 \qquad \Rightarrow \qquad \mathbf{E} = 0. \tag{6.14}$$

Thus on a spray m space with $\mathbf{B} = 0$, the S-curvature is always a 1-form.

Example 6.1.1 Consider the following special spray on an open subset $\mathcal{U} \subset \mathrm{R}^n$

$$\mathbf{G} := y^i \frac{\partial}{\partial x^i} - 2P(y) y^i \frac{\partial}{\partial y^i}, \tag{6.15}$$

where P satisfies

$$P(\lambda y) = \lambda P(y), \qquad \lambda > 0,\ y \in T_x\mathcal{U}.$$

The geodesics of $\mathbf{G}$ are straight lines in $\mathcal{U}$. A direct computation yields

$$\begin{aligned} B^i_{jkl} &= (Py^i)_{y^j y^k y^l} && (6.16)\\ &= P_{y^j y^k y^l} y^i + P_{y^j y^l}\delta^i_k + P_{y^j y^k}\delta^i_l + P_{y^k y^l}\delta^i_j. && (6.17)\end{aligned}$$

Then

$$\mathbf{E}_y(u,v) = \frac{1}{2} B^m_{ijm}(y) u^i v^j = \frac{n+1}{2} P_{y^i y^j} u^i v^j. \tag{6.18}$$

We can conclude that

$$\mathbf{B} = 0 \iff \mathbf{E} = 0.$$

Therefore, $\mathbf{G}$ is an affine spray if and only if P is a 1-form on $\mathcal{U}$. ♯

More general, consider two sprays $\mathbf{G}$ and $\tilde{\mathbf{G}}$ on a manifold M. Let

$$\mathbf{Q} := \frac{1}{2}\Big(\mathbf{G} - \tilde{\mathbf{G}}\Big).$$

$\mathbf{Q}$ is a vertical vector field on $TM \setminus \{0\}$. Take the vertical lift $\{e^v_i\}_{i=1}^n$ of a basis $\{e_i\}_{i=1}^n$ for T_xM and express $\mathbf{Q} = Q^i(y) e^v_i$. Q^i are positively homogeneous of degree two, namely, $Q^i(\lambda y) = \lambda^2 Q^i(y)$, $\forall \lambda > 0$. Let

$$Q^i_{jkl}(y) := \frac{\partial^3 Q^i}{\partial y^j \partial y^k \partial y^l}(y).$$

Define $\mathbf{Q}_y(u,v,w) := Q^i_{jkl}(y) u^j v^k w^l$, where $u = u^i e_i$, $v = v^j e_j$, $w = w^k e_k$. Let $\mathbf{B}$ and $\tilde{\mathbf{B}}$ denote the Berwald curvature of $\mathbf{G}$ and $\tilde{\mathbf{G}}$ respectively. A direction computation yields

$$\tilde{\mathbf{B}} = \mathbf{B} + \mathbf{Q}. \tag{6.19}$$

Consider a two-dimensional spray on an open subset $\mathcal{U} \subset \mathrm{R}^2$

$$\mathbf{G} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - 2G(x,y,u,v)\frac{\partial}{\partial u} - 2H(x,y,u,v)\frac{\partial}{\partial v}.$$

Write $G = G(x,y,u,v)$ and $H = H(x,y,u,v)$ in the following form

$$\begin{aligned} G &= \frac{1}{2}\Theta\Big(x,y,\frac{v}{u}\Big)\, u^2 \\ H &= \frac{1}{2}\Theta\Big(x,y,\frac{v}{u}\Big)\, uv - \frac{1}{2}\Phi\Big(x,y,\frac{v}{u}\Big)\, u^2, \end{aligned}$$

where $\Theta = \Phi(x, y, \xi)$ and $\Psi = \Psi(x, y, \xi)$ are C^∞ functions on $\mathcal{U} \times \mathrm{R}$. By a direct computation, we obtain a formula for $\mathbf{B}_{\mathbf{y}}(\mathbf{v}, \mathbf{v}, \mathbf{v}) = B^1 \frac{\partial}{\partial x} + B^2 \frac{\partial}{\partial y}$,

$$B^1 = -\frac{1}{2}\Theta_{\xi\xi\xi}\Big(\xi\mu - \nu\Big)^3 \tag{6.20}$$

$$B^2 = -\frac{1}{2}\Big\{\xi\Theta_{\xi\xi\xi} + \Big(3\Theta_{\xi\xi} - \Phi_{\xi\xi\xi}\Big)\Big\}\Big(\mu\xi - \nu\Big)^3, \tag{6.21}$$

where $\mathbf{y} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$, $\mathbf{v} = \mu\frac{\partial}{\partial x} + \nu\frac{\partial}{\partial y}$ and $\xi = v/u$. It follows from the above formula that $\mathbf{G}$ is affine if and only if

$$\Theta_{\xi\xi\xi} = 0, \qquad 3\Theta_{\xi\xi} - \Phi_{\xi\xi\xi} = 0. \tag{6.22}$$

By the above formula for $\mathbf{B}$, we immediately obtain a formula for $\mathbf{E}$

$$\mathbf{E}_{\mathbf{y}}(\mathbf{v}, \mathbf{v}) = \frac{1}{4}\Big(3\Theta_{\xi\xi} - \Phi_{\xi\xi\xi}\Big)\Big(\mu\xi - \nu\Big)^2. \tag{6.23}$$

Clearly, there exist two-dimensional sprays which are weakly affine, but not affine. For example, we can take

$$\Theta = \frac{1}{3}\Phi_\xi + A\xi + B,$$

where $A = A(x, y)$, $B = B(x, y)$ and $\Phi = \Phi(x, y, \xi)$ is an arbitrary function such that $\Phi_{\xi\xi\xi\xi} \neq 0$. For this set of $\{\Theta, \Phi\}$, the corresponding spray is weakly affine, but not affine.

Now we consider Finsler sprays. First, we make the following

Definition 6.1.4 A Finsler metric is called a *Berwald metric* if the induced spray is affine.

According to Z. Szabó [Sz1], if an affine spray is locally induced by a positive definite Finsler metric, then it is induced by a Riemannian metric. Based on this observation, Z. Szabó proved the following remarkable result.

Theorem 6.1.5 (Szabó) *Besides the Riemannian spaces and Minkowski spaces, there are exactly 54 kinds of irreducible and globally symmetric non-Riemannian Berwald spaces such that all other simply-connected and complete Berwald spaces can be globally decomposed to the Cartesian product of the above 56 spaces.*

Here a Finsler space is said to be *irreducible and globally symmetric* if the canonical affine connection (i.e., the Berwald connection) is irreducible and globally symmetric. The above theorem implies the following

Proposition 6.1.6 (Szabó) *Every positive definite two-dimensional Berwald metric is either locally Minkowskian or Riemannian.*

See [BaChSh1] for a proof. A brief argument is also given at the end of Section 11.1.

Two-dimensional singular Berwald metrics were classified by L. Berwald [Bw5]. He found all 2-dimensional Berwald metrics which are not locally Minkowskian. See also [Ma4] [AnInMa] and [MaOk].

Below are several two-dimensional conformally Minkowski Berwald metrics which have been studied by Antonelli and his collaborators in biology.

Example 6.1.2 ([AnHaMo]) Let $L = L(x, y, u, v)$ be a Finsler metric on an open subset $\mathcal{U} \subset \mathrm{R}^2$ in the form

$$L = \exp\Big[-2\alpha_1 x + 2(\lambda+1)\alpha_2 y + 2\nu_3 xy\Big] \cdot \Big(\frac{v^{1+\frac{1}{\lambda}}}{u^{\frac{1}{\lambda}}}\Big)^2, \tag{6.24}$$

where λ and α_i are positive constants. Antonelli-Han-Modayil compute the spray coefficients $G = G^1$ and $H = G^2$ of the spray induced by L

$$\begin{aligned} G &= \frac{1}{2}\lambda\Big(\alpha_1 - \nu_3 y\Big)u^2 \\ H &= \frac{1}{2}\Big(\alpha_2 + \frac{\nu_3}{\lambda+1}x\Big)v^2. \end{aligned}$$

Thus L is a Berwald metric which is also conformally Minkowski. See Example 8.2.6 below for further discussion. ♯

In higher dimensions, there are also many interesting singular Berwald metrics. The following examples are given by M. Matsumoto.

Example 6.1.3 ([Ma12]) Let $\rho = \rho(x, y, z)$ be a C^∞ function on an open subset $\Omega \subset \mathrm{R}^3$. Consider a conformally Minkowski metric

$$L_1 := e^{\rho(x,y,z)}\Big(uvw\Big)^{2/3}.$$

A direct computation gives

$$\det\Big(g_{ij}\Big) = \frac{1}{27}e^{3\rho(x,y,z)}.$$

The sign of g_{11} tells us that L_1 is not positive definite. The spray coefficients of L_1 are given by

$$G^1 = \frac{3}{4}\rho_x(x,y,z)\, u^2, \qquad G^2 = \frac{3}{4}\rho_y(x,y,z)\, v^2, \qquad G^3 = \frac{3}{4}\rho_z(x,y,z)\, w^2.$$

Thus L_1 is a Berwald metric.

Consider another Finsler metric.

$$L_2 := \Big[3a(x,z)uw^2 + b(y)v^3\Big]^{2/3}.$$

A direct computation gives

$$\det\Big(g_{ij}\Big) = -4\frac{a(x,y)^2 b(y)\ vw^2}{3a(x,z)uw^2 + b(y)v^3}.$$

Thus L_2 is not positive definite. The spray coefficients of L_2 are given by

$$G^1 = \frac{1}{2}\frac{a_x(x,z)}{a(x,z)}\ u^2, \qquad G^2 := \frac{1}{6}\frac{b_y(y)}{b(y)}\ v^2, \qquad G^3 := \frac{1}{4}\frac{a_z(x,z)}{a(x,z)}\ w^2.$$

Thus L_2 is a Berwald metric.

Finally, we consider the following Finsler metric

$$L_3 := e^{\rho(x,y,z)}\Big[u^3 + v^3 + w^3 - 3uvw\Big]^{2/3}.$$

A direct computation gives

$$\det\Big(g_{ij}\Big) = -e^{3\rho(x,y,z)}.$$

Thus L_3 is not positive definite. The spray coefficients of L_3 are given by

$$\begin{aligned} G^1 &= \frac{1}{4}\Big(\rho_x\ u^2 + \rho_z\ v^2 + \rho_y\ w^2 + 2\rho_y\ uv + 2\rho_z\ uw + 2\rho_x\ vw\Big) \\ G^2 &= \frac{1}{4}\Big(\rho_z\ u^2 + \rho_y\ v^2 + \rho_x\ w^2 + 2\rho_x\ uv + 2\rho_y\ uw + 2\rho_z\ vw\Big) \\ G^3 &= \frac{1}{4}\Big(\rho_y\ u^2 + \rho_x\ v^2 + \rho_z\ w^2 + 2\rho_z\ uv + 2\rho_x\ uw + 2\rho_y\ vw\Big) \end{aligned}$$

Thus L_3 is also a Berwald metric. ♯

6.2 Landsberg Curvature of Finsler Metrics

Let (M, L) be a Finsler space. For a tangent vector $y \in T_xM$, define

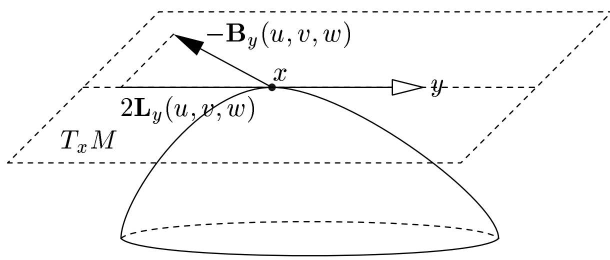
$$\mathbf{L}_y(u,v,w) := -\frac{1}{2}g_y(\mathbf{B}_y(u,v,w),y). \tag{6.25}$$

In local coordinates,

$$\mathbf{L}_y(u,v,w) := L_{ijk}(y)u^iv^jw^k, \tag{6.26}$$

where $u = u^i\frac{\partial}{\partial x^i}|_x, v = v^j\frac{\partial}{\partial x^j}|_x, w = w^k\frac{\partial}{\partial x^k}|_x$ and

$$L_{ijk}(y) := -\frac{1}{2}y^m g_{ml}(y)B^l_{ijk}(y) = -\frac{1}{2}y^m g_{ml}(y)\frac{\partial^3 G^l}{\partial y^i \partial y^j \partial y^k}(y). \tag{6.27}$$



$\mathbf{L}_y$ is a symmetric multi-linear form. It follows from (6.6) and (6.25) that

$$\mathbf{L}_y(y, v, w) = 0. \tag{6.28}$$

Definition 6.2.1 ([La1][La2]) **L** is called the *Landsberg curvature.* A Finsler metric is called a *Landsberg metric* if $\mathbf{L} = 0$.

It was L. Berwald [Bw2] who first called Finsler metrics with $\mathbf{L} = 0$ the Landsberg metrics. From (6.25), we see that

$$\mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{L} = 0. \tag{6.29}$$

Thus, Berwald metrics are always Landsberg metrics.

For simplicity, we let

$$\begin{aligned} g_{ij;k} : &= \frac{\partial g_{ij}}{\partial x^k} - g_{im}\Gamma^m_{jk} - g_{jm}\Gamma^m_{ik} - 2C_{ijm}N^m_k \\ C_{ijk;l} : &= \frac{\partial C_{ijk}}{\partial x^l} - N^m_l \frac{\partial C_{ijk}}{\partial y^m} - C_{mjk}\Gamma^m_{il} - C_{imk}\Gamma^m_{jl} - C_{ijm}\Gamma^m_{kl}. \end{aligned}$$

The reader will see that $g_{ij;k}$ and $C_{ijk;l}$ are the coefficients of the *horizontal covariant derivatives* of g_{ij} and C_{ijk}. See Chapters 9 and 10 below. The following lemma is useful.

Lemma 6.2.2 *The Landsberg curvature coefficients L_{ijk} are given by the following expressions.*

$$L_{ijk} = -\frac{1}{2} g_{ij;k} \tag{6.30}$$

$$L_{ijk} = C_{ijk;m} y^m. \tag{6.31}$$

Proof: According to (4.38), it suffices to prove (6.31). Let

$$C_{ijk\cdot l} := \frac{\partial C_{ijk}}{\partial y^l}, \qquad C_{ijk\cdot l\cdot m} := \frac{\partial^2 C_{ijk}}{\partial y^l \partial y^m}.$$

The reader will see that $C_{ijk\cdot l}$ are the coefficients of the *vertical covariant derivates* of C_{ijk} and etc. See Chapters 9 and 10 below. Differentiating (4.41) with respect to y^m yields

$$\begin{aligned} g_{jl}\frac{\partial^3 G^l}{\partial y^i \partial y^k \partial y^m} &= y^l \frac{\partial C_{ijk\cdot m}}{\partial x^l} - 2G^l C_{ijk\cdot l\cdot m} \\ &\quad -2\Big\{ C_{ijl}\Gamma^l_{km} + C_{jkl}\Gamma^l_{im} + C_{jlm}\Gamma^l_{ik}\Big\} \\ &\quad -2\Big\{ C_{jkl\cdot m}N^l_i + C_{ijl\cdot m}N^l_k + C_{ijk\cdot l}N^l_m\Big\} \\ &\quad +\frac{\partial C_{jkm}}{\partial x^i} + \frac{\partial C_{ijm}}{\partial x^k} + \frac{\partial C_{ijk}}{\partial x^m} - \frac{\partial C_{ikm}}{\partial x^j}. \end{aligned} \tag{6.32}$$

By the homogeneity of L, we have

$$y^j C_{ijk\cdot m} = -C_{ikm}, \qquad y^j C_{ijk\cdot l\cdot m} = -2C_{ikl\cdot m}. \tag{6.33}$$

Contracting (6.32) with $\frac{1}{2}y^j$ and using (6.33), we immediately obtain

$$\frac{1}{2}y^j g_{jl}\frac{\partial^3 G^l}{\partial y^i \partial y^k \partial y^m} = -y^j \frac{\partial C_{ikm}}{\partial x^j} + 2G^l C_{ikl\cdot m} + C_{ikl}N^l_m + C_{ilm}N^l_k + C_{klm}N^l_i.$$

This completes the proof. Q.E.D.

For simplicity, let

$$L_{ijk\cdot l} := \frac{\partial L_{ijk}}{\partial y^l}.$$

$L_{ijk\cdot l}$ are the coefficients of the *vertical covariant derivatives* of L_{ijk}.

Lemma 6.2.3 *The Berwald curvature is determined by the Cartan torsion and the Landsberg curvature. More precisely,*

$$B^i_{jkl} = g^{im}\Big\{ C_{mjl;k} + C_{mkl;j} - C_{jkl;m} + L_{mjk\cdot l}\Big\}. \tag{6.34}$$

The reader can verify (6.34) directly. But there is a much simpler way to prove (6.34) using the exterior differentiation method that will be given later. See (10.19) below. So we omit the technical proof here.

It follows from Lemmas 6.2.2 and 6.2.3 that if $\mathbf{C} = 0$, then $\mathbf{L} = 0$ and $\mathbf{B} = 0$. In this case, the Finsler metric must be Riemannian.

Recall the definition of the mean Cartan torsion $\mathbf{I}_y : T_xM \to \mathrm{R}$

$$\mathbf{I}_y(v) := \sum_{i,j=1}^n \mathbf{C}_y(e_i, e_j, v) g^{ij}(y), \tag{6.35}$$

where $\{e_i\}_{i=1}^n$ is an arbitrary basis for T_xM and $g_{ij}(y) := g_y(e_i, e_j)$. Similarly, we define $\mathbf{J}_y : T_xM \to \mathrm{R}$ by

$$\mathbf{J}_y(v) := \sum_{i,j=1}^{n} \mathbf{L}_y(e_i, e_j, v) g^{ij}(y), \tag{6.36}$$

We call $\mathbf{J}$ the *mean Landsberg curvature*. The mean Landsberg curvature is an important geometric quantity in Finsler geometry.

Take a standard local coordinate system (x^i, y^i) in TM. Let

$$I_i(y) := \mathbf{I}_y\Big(\frac{\partial}{\partial x^i}|_x\Big), \qquad J_i(y) := \mathbf{J}_y\Big(\frac{\partial}{\partial x^i}|_x\Big).$$

I_i and J_i can be expressed by

$$I_i = g^{jk} C_{ijk}, \qquad J_i := g^{jk} C_{ijk}.$$

Let

$$E_{ij}(y) := \mathbf{E}_y\Big(\frac{\partial}{\partial x^i}|_x, \frac{\partial}{\partial x^j}|_x\Big).$$

E_{ij} can be expressed by

$$E_{ij} = \frac{1}{2} B^m_{ijm}.$$

For simplicity, let

$$I_{i;j} := \frac{\partial I_i}{\partial x^j} - I_m \Gamma^m_{ij} - \frac{\partial I_i}{\partial y^m} N^m_j, \qquad J_{i\cdot j} := \frac{\partial J_i}{\partial y^j}.$$

The following lemma is useful

Lemma 6.2.4 *The Landsberg curvature is determined by the mean Cartan torsion. The mean Berwald curvature is determined by the mean Cartan torsion and the mean Landsberg curvature. More precisely,*

$$J_i = I_{i;k} y^k \tag{6.37}$$

$$E_{ij} = \frac{1}{2}\Big\{ I_{j;i} + J_{i\cdot j} \Big\}. \tag{6.38}$$

Proof: (6.37) follows from (6.31). (6.38) follows from (6.34). We will prove Lemma 6.2.4 later using exterior differentiation method. See (10.27) and (10.28) below.

Q.E.D.

It follows from Lemma 6.2.4 that if $\mathbf{I} = 0$, then $\mathbf{J} = 0$ and $\mathbf{E} = 0$. But the converse is not true. There are many Berwald metrics with $\mathbf{I} \neq 0$. We will present many such examples.

Let F be a positive definite Finsler metric on M. Let $\hat{g}$ and $\dot{g}$ the Sasaki metric of F on $TM \setminus \{0\}$ and SM respectively (see (5.14)). It is proved in [Ai] that if F is a Landsberg metric, then every $T_xM \setminus \{0\}$ is totally geodesic in $(TM \setminus \{0\}, \hat{g})$. Hence, every S_x is totally geodesic in $(SM, \dot{g})$. The author proved (unpublished) that if $\mathbf{J} = 0$, then every $T_xM \setminus \{0\}$ is minimal in $(TM \setminus \{0\}, \hat{g})$ and every S_xM is minimal in $(SM, \dot{g})$. This implies that the volume function of the unit tangent sphere S_xM

$$V(x) := \mu_{\dot{g}_x}(\mathrm{S}_xM)$$

is a constant, where $\dot{g}_x$ denotes the induced Riemannian metric on S_xM. This fact is due to D. Bao and Z. Shen [BaSh1].

Example 6.2.1 Consider a Randers metric $F := \alpha+\beta$, where $\alpha(y) = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta(y) = b_i(x)y^i$ is a 1-form on M. Assume that $\|\beta\| := \sup_{\alpha(y)=1} \beta(y) < 1$. This condition guarantees that F is a positive definite Finsler metric. M. Matsumoto [Ma2] proved that F is a Landsberg metric if and only if β is parallel with respect to α. Namely,

$$b_{i;j} = \frac{\partial b^i}{\partial x^j} - \gamma^k_{ij} b^k = 0.$$

From (5.28), we see that if β is parallel, then the spray of F coincides with that of α. Thus F is actually a Berwald metric [HaIc1]. In [SSAY], Shibata-Shimada-Azuma-Yasuda reproved Matsumoto's theorem. Later on, S. Kikuchi [Ki] proved that a Randers metric $F = \alpha + \beta$ is a Berwald metric if and only if β is parallel.

Let $(\bar{M}, \bar{\alpha})$ be an arbitrary Riemannian space of dimension $n-1$ and $M := \bar{M} \times \mathrm{S}^1$ the product space with the standard product metric

$$\alpha(y) := \sqrt{[\bar{\alpha}(\bar{y})]^2 + s^2}, \qquad y = (\bar{y}, s) \in T_{(\bar{x},t)}(\bar{M} \times \mathrm{S}^1).$$

The 1-form $\beta := \varepsilon dt$, where $|\varepsilon| < 1$, is parallel with respect to α. Thus

$$F(y) := \alpha(y) + \beta(y)$$

is a positive definite Berwald metric on M. ♯

Example 6.2.2 Let F denote the Funk metric on a strongly convex domain Ω in R^n. According to Section 4.3, the spray coefficients of F are given by

$$G^i(y) = \frac{1}{2}F(y)y^i.$$

We are going to compute the Berwald curvature and the Landsberg curvature for F. It follows from (6.16) and (6.18) that

$$B^i_{jkl} = \frac{1}{2}(Fy^i)_{y^jy^ky^l}, \quad E_{ij} = \frac{n+1}{4}F_{y^iy^j}. \tag{6.39}$$

Let

$$h_{ij} := F\,F_{y^iy^j} = g_{ij} - \frac{1}{F^2}g_{is}g_{jt}y^sy^t.$$

For a vector $y \in T_xM \setminus \{0\}$, define

$$h_y(u,v) := h_{ij}(y)u^iv^j,$$

where $u = u^i\frac{\partial}{\partial x^i}|_x, v = v^j\frac{\partial}{\partial x^j}|_x$. h is called the *angular metric tensor.*

A direct computation yields

$$\begin{aligned}
\mathbf{B}_y(u,v,w) &= -\frac{1}{2F^3(y)}\Big\{h_y(u,v)g_y(w,y) + h_y(v,w)g_y(u,y) \\
&\quad + h_y(u,w)g_y(v,y)\Big\}y \\
&\quad + \frac{1}{2F(y)}\Big\{h_y(u,v)w + h_y(v,w)u + h_y(u,w)v\Big\} \\
&\quad + \frac{1}{F(y)}\mathbf{C}_y(u,v,w)\,y, \\
\mathbf{E}_y(u,v) &= \frac{n+1}{4F(y)}h_y(u,v).
\end{aligned}$$

Thus

$$\mathbf{L}_y(u,v,w) := -\frac{1}{2}g_y(\mathbf{B}_y(u,v,w),y) = -\frac{F(y)}{2}\mathbf{C}_y(u,v,w).$$

Finally, we obtain

$$\mathbf{L} = -\frac{F}{2}\mathbf{C}, \qquad \mathbf{E} = \frac{n+1}{4F}h. \tag{6.40}$$

♯

6.3 Finsler Surfaces

Now we study the mean Berwald curvature and the mean Landsberg curvature of Finsler surfaces. See (6.10) and (6.36) for definitions. For simplicity, we will only consider positive definite Finsler surfaces. Let F be a Finsler metric on a surface M. For a vector $\mathbf{y} \in T_xM \setminus \{0\}$, let $\{\mathbf{y}, \mathbf{y}^\perp\}$ be the Berwald basis for T_xM which is defined by (1.31). By the definitions of $\mathbf{E}$, $\mathbf{L}$ and $\mathbf{J}$, we have

$$\begin{aligned}
\mathbf{J}_\mathbf{y}(\mathbf{y}^\perp) &= \frac{1}{F^2(\mathbf{y})}\mathbf{L}_\mathbf{y}(\mathbf{y}^\perp,\mathbf{y}^\perp,\mathbf{y}^\perp) \\
&= -\frac{1}{2F^2(\mathbf{y})}g_\mathbf{y}(\mathbf{B}_\mathbf{y}(\mathbf{y}^\perp,\mathbf{y}^\perp,\mathbf{y}^\perp),\mathbf{y}) \\
\mathbf{E}_\mathbf{y}(\mathbf{y}^\perp,\mathbf{y}^\perp) &= \frac{1}{2F^2(\mathbf{y})}g_\mathbf{y}(\mathbf{B}_\mathbf{y}(\mathbf{y}^\perp,\mathbf{y}^\perp,\mathbf{y}^\perp),\mathbf{y}^\perp).
\end{aligned}$$

For simplicity, let

$$\mathbf{B}(\mathbf{y}) := F^{-1}(\mathbf{y})\mathbf{B}_{\mathbf{y}}(\mathbf{y}^\perp, \mathbf{y}^\perp, \mathbf{y}^\perp) \tag{6.41}$$

$$\mathbf{J}(\mathbf{y}) := F^{-1}(\mathbf{y})\mathbf{J}_{\mathbf{y}}(\mathbf{y}^\perp), \tag{6.42}$$

$$\mathbf{E}(\mathbf{y}) := F^{-1}(\mathbf{y})\mathbf{E}_{\mathbf{y}}(\mathbf{y}^\perp, \mathbf{y}^\perp). \tag{6.43}$$

It is easy to see that

$$\mathbf{B}(\mathbf{y}) = -2\mathbf{J}(\mathbf{y})\,\mathbf{y} + 2\mathbf{E}(\mathbf{y})\,\mathbf{y}^\perp.$$

Thus, $\mathbf{B}(\mathbf{y})$ is completely determined by $\mathbf{J}(\mathbf{y})$ and $\mathbf{E}(\mathbf{y})$. We immediately obtain the following

Proposition 6.3.1 *A two-dimensional Finsler metric is Berwaldian if and only if it is Landsbergian and weakly Berwaldian.*

Now we give the local formulas for $\mathbf{E}(\mathbf{y})$ and $\mathbf{J}(\mathbf{y})$. Let (x, y) denote the local coordinates in M and (x, y, u, v) the standard local coordinate system in TM. Let

$$L(x, y, u, v) := F^2\Big(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\Big).$$

Since F is positive definite,

$$L_{uu}L_{vv} - L_{uv}L_{uv} > 0. \tag{6.44}$$

By (1.32), for a vector $\mathbf{y} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \in T_{(x,y)}M$, the vector $\mathbf{y}^\perp$ orthogonal to $\mathbf{y}$ is given by

$$\mathbf{y}^\perp = \frac{-L_v\frac{\partial}{\partial x} + L_u\frac{\partial}{\partial y}}{\sqrt{L_{uu}L_{vv} - L_{uv}L_{uv}}}.$$

$\mathbf{y}^\perp$ has the following property:

$$g_{\mathbf{y}}(\mathbf{y}, \mathbf{y}^\perp) = 0, \qquad g_{\mathbf{y}}(\mathbf{y}^\perp, \mathbf{y}^\perp) = F^2(\mathbf{y}).$$

Express the spray of F by

$$\mathbf{G} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - 2G(x, y, u, v)\frac{\partial}{\partial u} - 2H(x, y, u, v)\frac{\partial}{\partial v}.$$

By (4.34) and (4.35), we obtain

$$G := \frac{L_{vv}(L_x u + L_y v) - L_v(L_{xv}u + L_{yv}v - L_y)}{2(2LL_{vv} - L_vL_v)}\,u, \tag{6.45}$$

$$H := \frac{v}{u}G - \frac{L_v(L_x u + L_y v) - 2L(L_{xv}u + L_{yv}v - L_y)}{2(2LL_{vv} - L_vL_v)}. \tag{6.46}$$

It follows from the homogeneity of G and H that

$$G_{uvv} = -\frac{v}{u}G_{vvv}, \qquad G_{uuv} = \Big(\frac{v}{u}\Big)^2 G_{vvv}, \qquad G_{uuu} = -\Big(\frac{v}{u}\Big)^3 G_{vvv}.$$

The similar identities hold for H. By these identities, we obtain

$$\mathbf{B}(\mathbf{y}) = \frac{8F^5(\mathbf{y})}{u^3} \frac{G_{vvv}\frac{\partial}{\partial x} + H_{vvv}\frac{\partial}{\partial y}}{\Big(L_{uu}L_{vv} - L_{uv}L_{uv}\Big)^{\frac{3}{2}}}. \tag{6.47}$$

This gives

$$\mathbf{J}(\mathbf{y}) = -\frac{2F^3(\mathbf{y})}{u^3}\frac{G_{vvv}L_u + H_{vvv}L_v}{\Big(L_{uu}L_{vv} - L_{uv}L_{uv}\Big)^{\frac{3}{2}}} \tag{6.48}$$

$$\mathbf{E}(\mathbf{y}) = -\frac{2F^3(\mathbf{y})}{u^3}\frac{G_{vvv}v - H_{vvv}u}{L_{uu}L_{vv} - L_{uv}L_{uv}}. \tag{6.49}$$

Express

$$F(x,y,u,v) = \phi\Big(x,y,\frac{v}{u}\Big)\, u, \qquad u > 0, \tag{6.50}$$

where $\phi = \phi(x,y,\xi) > 0$ is a C^∞ function. Note that

$$L_{uu}L_{vv} - L_{uv}L_{uv} = 4\phi^3\phi_{\xi\xi}.$$

Thus (6.44) is equivalent to that $\phi_{\xi\xi} > 0$. From (6.45) and (6.46), we obtain

$$G = \frac{1}{2}\Theta\Big(x,y,\frac{v}{u}\Big)\, u^2$$

$$H = \frac{1}{2}\Theta\Big(x,y,\frac{v}{u}\Big)\, uv - \frac{1}{2}\Phi\Big(x,y,\frac{v}{u}\Big)\, u^2,$$

where

$$\Phi = \frac{\phi_y - \phi_{x\xi} - \phi_{y\xi}\xi}{\phi_{\xi\xi}}, \tag{6.51}$$

$$\Theta = \Phi\frac{\phi_\xi}{\phi} + \frac{\phi_x + \phi_y\xi}{\phi}. \tag{6.52}$$

Differentiating G and H gives

$$G_{vvv} = \frac{1}{2u}\Theta_{\xi\xi\xi}, \qquad H_{vvv} = \frac{1}{2u}\Big[3\Theta_{\xi\xi} + \Theta_{\xi\xi\xi}\xi - \Phi_{\xi\xi\xi}\Big].$$

Plugging them into (6.48) and (6.49) yields

$$\mathbf{J}(\mathbf{y}) = -\frac{1}{4\phi}\Big[\Theta_{\xi\xi\xi} + \Big(3\Theta_{\xi\xi} - \Phi_{\xi\xi\xi}\Big)\frac{\phi_\xi}{\phi}\Big]\Big(\frac{\phi}{\phi_{\xi\xi}}\Big)^{\frac{3}{2}}, \tag{6.53}$$

$$\mathbf{E}(\mathbf{y}) = \frac{1}{4\phi}\Big[3\Theta_{\xi\xi} - \Phi_{\xi\xi\xi}\Big]\frac{\phi}{\phi_{\xi\xi}}. \tag{6.54}$$

Thus $\mathbf{J} = 0$ if and only if

$$\Theta_{\xi\xi\xi} + \Big(3\Theta_{\xi\xi} - \Phi_{\xi\xi\xi}\Big)\frac{\phi_\xi}{\phi} = 0 \tag{6.55}$$

and $\mathbf{E} = 0$ if and only if

$$3\Theta_{\xi\xi} - \Phi_{\xi\xi\xi} = 0. \tag{6.56}$$

It follows from (6.55) and (6.56) that $\mathbf{B} = 0$ if and only if

$$\Theta_{\xi\xi\xi} = 0, \qquad 3\Theta_{\xi\xi} - \Phi_{\xi\xi\xi} = 0. \tag{6.57}$$

Example 6.3.1 Consider the following Finsler metric F on an open subset $\mathcal{U} \subset \mathrm{R}^2$:

$$F(x, y, u, v) := \Big\{u^4 + 2c(x, y)u^2v^2 + v^4\Big\}^{\frac{1}{4}}, \tag{6.58}$$

where $0 < c(x, y) < 3$. The formula for $\mathbf{J}$ is given by

$$\begin{aligned}\mathbf{J} \;\; = \;\; & \chi\Big\{(c^3 c_x - 5cc_x)\, u^5 - 4cc_y\, u^4 v + (3c_x - 3c^2 c_x)\, u^3 v^2 \\ & +(3c_y - 3c^2 c_y)\, u^2 v^3 + 4cc_x\, uv^4 - (c^3 c_y - 5cc_y)\, v^5\Big\},\end{aligned} \tag{6.59}$$

where

$$\chi = \frac{3uv}{2} \frac{\Big(u^4 + 2cu^2v^2 + v^4\Big)^{\frac{3}{2}}}{\Big(cu^4 + (3 - c^2)u^2v^2 + cv^4\Big)^{\frac{7}{2}}}.$$

Thus $\mathbf{J} = 0$ if and only if $c(x, y) = constant$, i.e., F is a Minkowski functional. The formula for $\mathbf{E}$ is much more complicated, so is omitted. ♯

A natural question is whether or not there is a Landsberg metric which is not a Berwald metric. From the above argument, we conclude that if ϕ satisfies

$$\Theta_{\xi\xi\xi} + \Big(3\Theta_{\xi\xi} - \Phi_{\xi\xi\xi}\Big)\frac{\phi_\xi}{\phi} = 0, \quad \Theta_{\xi\xi\xi} \neq 0, \tag{6.60}$$

then $F = u\phi\Big(x, y, \dfrac{v}{u}\Big)$ is a Landsberg metric, but not a Berwald metric.

In order to find a Landsberg metric, one may try to solve (6.55). Let us expand the equation (6.55) in terms of ϕ.

$$\begin{aligned}&\Big(\phi\phi_{\xi\xi}\phi_{\xi\xi}\phi_{\xi\xi\xi} - 3\phi_\xi\phi_{\xi\xi}\phi_{\xi\xi}\phi_{\xi\xi}\Big)\Big(\phi_x + \phi_y\xi\Big) + \Big(3\phi_\xi\phi_\xi\phi_{\xi\xi}\phi_{\xi\xi} \\ &-2\phi\phi\phi_{\xi\xi}\phi_{\xi\xi\xi\xi} + 3\phi\phi\phi_{\xi\xi\xi}\phi_{\xi\xi\xi} + 2\phi\phi_\xi\phi_{\xi\xi}\phi_{\xi\xi\xi}\Big)\Big(\phi_{x\xi} + \phi_{y\xi}\xi - \phi_y\Big) \\ &-\Big(3\phi\phi_\xi\phi_{\xi\xi}\phi_{\xi\xi} + 3\phi\phi\phi_{\xi\xi}\phi_{\xi\xi\xi}\Big)\Big(\phi_{x\xi\xi} + \phi_{y\xi\xi\xi}\xi\Big) \\ &+\Big(2\phi\phi\phi_{\xi\xi}\phi_{\xi\xi}\Big)\phi_{x\xi\xi\xi} + \Big(6\phi\phi_{\xi\xi}\phi_{\xi\xi}\phi_{\xi\xi}\Big)\phi_y + \Big(2\phi\phi\phi_{\xi\xi}\phi_{\xi\xi}\Big)\phi_{y\xi\xi}\xi = 0.\end{aligned}$$

Note that

$$\phi(x, y, \xi) = \sqrt{a + 2b\xi + c\xi^2}$$

is a solution of (6.57), where $a = a(x, y) > 0, b = b(x, y) > 0$ and $c = c(x, y) > 0$ are C^∞ functions on R^2 satisfying $ac - b^2 > 0$. In this case, $F = u\phi\Big(x, y, \frac{v}{u}\Big)$ is a Riemannian metric.

Another solution is

$$\phi(x, y, \xi) = e^\rho \xi^{1+\lambda}, \tag{6.61}$$

where $\lambda > 0$ and $\rho = \rho(x, y)$ is a C^∞ function on R^2. In this case, $F = u\phi\Big(x, y, \frac{v}{u}\Big)$ is a Berwald metric.

There are many other solutions to (6.57). So far all the known solutions to (6.57) which can be expressed in terms of elementary functions are also the solutions to (6.56). Thus the resulting metrics $F = u\phi\Big(x, y, \frac{v}{u}\Big)$ are Berwald metrics.

Chapter 7

Connections

It was L. Berwald [Bw1] who first introduced the notion of connection for Finsler metrics. After his remarkable work, several connections were introduced from various approaches. The well-known connections are the Cartan connection [Ca] and the Chern connection [Ch1] [BaCh]. They all differ from the Berwald connection by the Cartan torsion, or the Landsberg curvature, or their linear combination. But only the Berwald connection can be extended to sprays! Therefore, we will mainly discuss the Berwald connection throughout this book. One is referred to [BaChSh1][Ma4][Sh1] for other connections.

7.1 Berwald Connection of Sprays

Roughly speaking, a connection on a manifold is a first order directional operator acting on vector fields. The Euclidean connection on an open subset $\mathcal{U}$ in the Euclidean space $\mathbb{R}^n$ is described as follows

$$\mathrm{D}_u V|_x = \Big(u^i \frac{\partial V^1}{\partial x^i}(x), \cdots, u^i \frac{\partial V^n}{\partial x^i}(x)\Big),$$

where $V = (V^1, \cdots, V^n) : \mathcal{U} \to \mathrm{R}^n$ is a C^∞ vector field on $\mathcal{U}$ and $u = (u^1, \cdots, u^n) \in T_x\mathcal{U} = \mathrm{R}^n$ is a tangent vector at x. The notion of the Euclidean connection can be extended to spray spaces.

Definition 7.1.1 A *connection* ∇ on M is a family of maps

$$\nabla := \Big\{\nabla^y : T_xM \times C^\infty(TM) \to T_xM, \qquad y \in T_xM \setminus \{0\},\ x \in M\Big\}$$

which has the following properties

(a) $\nabla_u^{\lambda y} V = \nabla_u^y V$;

(b) $\nabla_u^y (fV) = u(f)V + f\nabla_u^y V$;

(c) $\nabla^y_u(U+V) = \nabla^y_u U + \nabla^y_u V$;

(d) $\nabla^y_{fu} V = f\nabla^y_u V$;

(e) $\nabla^y_{u+v} V = \nabla^y_u V + \nabla^y_v V$;

(f) $\nabla^Y_U V - \nabla^Y_V U = [U, V]$;

where $\lambda > 0$, $u, v \in T_xM$ and $Y, U, V \in C^\infty(TM)$. Moreover, for any C^∞ vector fields $Y \neq 0$, U and V on M, the vector field $\nabla^Y_U V$ is C^∞.

Let ∇ be a connection on a manifold M. In a standard local coordinate system (x^i, y^i) in TM, define a set of local functions $\Gamma^i_{jk}(y)$ on TM by

$$\Gamma^i_{jk}(y)\frac{\partial}{\partial x^i}|_x := \nabla^y_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k}, \qquad y \in T_xM.$$

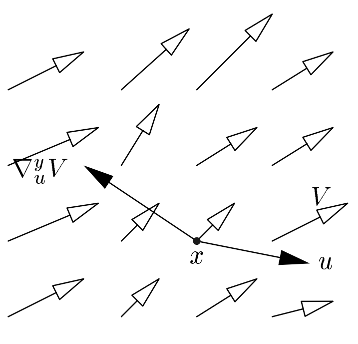
The last condition (f) implies

$$\Gamma^i_{jk} = \Gamma^i_{kj}.$$

In this sense, ∇ is *torsion-free*. For any $u = u^i\frac{\partial}{\partial x^i}|_x$ and $V = V^i\frac{\partial}{\partial x^i}$, $\nabla^y_u V$ is expressed by

$$\nabla^y_u V = \Big\{u(V^i)(x) + V^j(x)\Gamma^i_{jk}(y)u^k\Big\}\frac{\partial}{\partial x^i}|_x.$$

When ∇^y is independent of $y \in TM \setminus \{0\}$, ∇ is called an *affine connection* by H. Weyl.



There is a canonical connection for every spray. Let

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}$$

be a spray on a manifold M. The Christoffel symbols of $\mathbf{G}$ are defined by

$$\Gamma^i_{jk}(y) := \frac{\partial^2 G^i}{\partial y^j \partial y^k}(y). \tag{7.1}$$

Here $\Gamma^i_{jk}(y)$ are local functions on $TM\setminus\{0\}$. For a vector $y\in T_xM\setminus\{0\}$, define a map $\nabla^y : T_xM\times C^\infty(TM)\to T_xM$ by

$$\nabla^y_u V := \Big\{u(V^i)(x) + V^j(x)\Gamma^i_{jk}(y)u^k\Big\}\frac{\partial}{\partial x^i}\Big|_x, \tag{7.2}$$

where $u = u^i\frac{\partial}{\partial x^i}\Big|_x \in T_xM$ and $V = V^i\frac{\partial}{\partial x^i}\in C^\infty(TM)$. Clearly, $\nabla = \{\nabla^y\}_{y\in TM\setminus\{0\}}$ is a connection on M. We call it the *Berwald connection* of $\mathbf{G}$. From (7.1), we see that $\Gamma^i_{jk}=\Gamma^i_{kj}$. Thus the Berwald connection is always torsion-free. See [Bw1].

When $\mathbf{G}$ is an affine spray, i.e., there are local functions $\Gamma^i_{jk}(x)$ on M such that

$$G^i(y) = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$$

are quadratic in $y\in T_xM$ at every point $x\in M$, the Berwald connection is affine. We immediately obtain the following

Proposition 7.1.2 *The Berwald connection of a spray* $\mathbf{G}$ *is an affine connection on* M *if and only if* $\mathbf{B}=0$.

When a spray is induced by a Riemannian metric g, the Berwald connection is torsion-free and affine. We call it the *Levi-Civita* connection of g.

Example 7.1.1 Let (V, F) be a Minkowski space. F induces a Riemannian metric $\hat{g}$ on $\mathrm{V}\setminus\{0\}$ by

$$\hat{g}(u,v) := g_y(u,v), \qquad u,v\in T_y\mathrm{V}\approx \mathrm{V}. \tag{7.3}$$

Let $\hat{\mathrm{D}}$ denote the Levi-Civita connection of $\hat{g}$. Let (y^i) be a global coordinate system for V determined by a basis and Γ^i_{jk} the Christoffel symbols of $\hat{g}$. Observe that

$$\begin{aligned}\Gamma^i_{jk}(y) &= \frac{1}{2}g^{il}(y)\Big\{\frac{\partial g_{lk}}{\partial y^j}(y) + \frac{\partial g_{jl}}{\partial y^k}(y) - \frac{\partial g_{jk}}{\partial y^l}(y)\Big\}\\ &= g^{il}(y)\Big\{C_{jkl}(y) + C_{jkl}(y) - C_{jkl}(y)\Big\} \qquad (7.4)\\ &= C^i_{jk}(y), \qquad (7.5)\end{aligned}$$

where $C^i_{jk}(y) := g^{im}(y)C_{jkm}(y)$. ♯

For a spray space $(M,\mathbf{G})$, the core part of the Berwald connection is the so-called *N-connection*. It is a family of maps

$$\mathrm{D}_y : C^\infty(TM)\to T_xM, \qquad y\in T_xM,$$

defined by

$$\mathrm{D}_y V := \nabla^y_y V, \qquad V \in C^\infty(TM). \tag{7.6}$$

If $y = 0$, we set $\mathrm{D}_y V := 0$. In local coordinates,

$$\mathrm{D}_y V := \Big\{ y(V^i)(x) + V^j(x) N^i_j(y) \Big\} \frac{\partial}{\partial x^i}\Big|_x, \tag{7.7}$$

where $N^i_j(y) := \Gamma^i_{jk}(y) y^k = \frac{\partial G^i}{\partial y^j}(y)$.

With the Berwald connection, we can define another family of maps

$$\tilde{\mathrm{D}}_y : C^\infty(TM) \to T_x M, \qquad y \in T_x M,$$

by

$$\tilde{\mathrm{D}}_y V := \nabla^V_y V, \qquad V \in C^\infty(TM). \tag{7.8}$$

When $V(x) = 0$, we set $\tilde{\mathrm{D}}_y V := y(V^i)(x)$. In local coordinates,

$$\tilde{\mathrm{D}}_y V := \Big\{ y(V^i)(x) + y^j N^i_j(V) \Big\} \frac{\partial}{\partial x^i}\Big|_x. \tag{7.9}$$

Every Finsler metric L induces a spray $\mathbf{G}$. The Berwald connection of $\mathbf{G}$ is also called the *Berwald connection* of L. We have the following

Theorem 7.1.3 *On a Finsler space (M, L), the Berwald connection ∇ is the unique connection on TM satisfying*

$$\nabla^Y_U V - \nabla^Y_V U = [U, V], \tag{7.10}$$

$$\begin{aligned} W[g_Y(U,V)] &= g_Y(\nabla^Y_W U, V) + g_Y(U, \nabla^Y_W V) \\ &\quad + 2\mathbf{C}_Y(U, V, \nabla^Y_W Y) - 2\mathbf{L}_Y(U, V, W). \end{aligned} \tag{7.11}$$

where $U, V, W, Y \in C^\infty(TM)$ with $Y \neq 0$.

Proof: (7.10) is equivalent to that $\Gamma^i_{jk} = \Gamma^i_{kj}$ and (7.11) is equivalent to (6.30). Q.E.D.

Let Y be a non-zero vector field on an open subset $\mathcal{U}$. Y induces a smooth Riemannian metric on $\mathcal{U}$

$$\hat{g} := g_Y.$$

Let $\hat{\mathrm{D}}$ denote the Levi-Civita connection of $\hat{g}$. $\hat{\nabla} = \{\hat{\nabla}^y := \mathrm{D}\}$ the Berwald connection of the Finsler metric $\hat{L}(y) := \sqrt{\hat{g}(y,y)}$. It follows from (7.10) and (7.11) that

$$\begin{aligned} g_Y(\nabla^y_u V, W) &= \hat{g}(\hat{\nabla}^y_u V, W) \\ &\quad - \mathbf{C}_Y(V, W, \nabla^y_u Y) - \mathbf{C}_Y(U, W, \nabla^Y_V Y) \\ &\quad + \mathbf{C}_Y(V, U, \nabla^Y_W Y) + \mathbf{L}_Y(U, V, W). \end{aligned} \tag{7.12}$$

M. Matsumoto [Ma5][Ma7] has made some investigations on the Riemannian metric g_Y for a general vector field Y. In [Sh2], the author shows that the Riemannian metrics g_Y are extremely important, when Y is *geodesic*. Recall that a vector Y is said to be geodesic if

$$\mathrm{D}_Y Y = \nabla^Y_Y Y = 0.$$

By (7.12), we immediately obtain the following

Lemma 7.1.4 *Let (M, L) be a Finsler space and Y a non-zero geodesic field on an open subset $\mathcal{U}$ with $y = Y_x$. Then for any $V \in C^\infty(T\mathcal{U})$ with $v = V_x$ and $u \in T_xM$,*

$$\mathrm{D}_y V = \hat{\mathrm{D}}_y V, \tag{7.13}$$

$$\tilde{\mathrm{D}}_v Y = \hat{\mathrm{D}}_v Y, \tag{7.14}$$

$$g_y(\nabla^y_u V, y) = \hat{g}(\hat{\mathrm{D}}_u V, y), \tag{7.15}$$

where $\hat{\mathrm{D}}$ denotes the Levi-Civita connection of $\hat{g} := g_Y$. Hence Y is also a geodesic with respect to $\hat{g}$.

7.2 Chern Connection of Finsler Metrics

In the previous section, we introduced the Berwald connection for sprays. Since every Finsler metric induces a spray, the Berwald connection is also defined for Finsler metrics.

In 1934, E. Cartan became interested in Finsler spaces [Ca]. It was the fashion then that connections be metric-compatible such as the Levi-Civita connection in Riemannian geometry. So Cartan modified the Berwald connection for Finsler spaces and introduced the *Cartan connection*. The Cartan connection is indeed metric-compatible, but not torsion-free. Moreover it can be defined only on an appropriate vector bundle (e.g. the vertical tangent bundle) over $TM \setminus \{0\}$.

In 1943, S. S. Chern found another notable connection for Finsler metrics [Ch1][Ch3]. The Chern connection is different from the Berwald connection only by a quantity — the Landsberg curvature. The Chern connection is also called the *Rund connection* in literatures, because Chern's papers were written using Cartan's exterior differential methods which were not known to other geometers in Finsler geometry. The curvatures of both the Chern connection and the Berwald connection are in very simple form, since they are all torsion-free. In this section, we will give a brief description of the Chern connection. See [BaCh][BaChSh1] for more details.

Let (M, L) be a Finsler space. For a vector $y \in T_xM \setminus \{0\}$, the Landsberg curvature $\mathbf{L}_y : T_xM \times T_xM \times T_xM \to \mathrm{R}$ determines a bilinear symmetric form

$\mathbf{L}_y : T_xM \otimes T_xM \to T_xM$ by

$$g_y(\mathbf{L}_y(u,v), w) = \mathbf{L}_y(u,v,w).$$

(6.28) implies

$$\mathbf{L}_y(y,v) = 0. \tag{7.16}$$

Define a map $\tilde{\nabla}^y : T_xM \times C^\infty(TM) \to T_xM$ by

$$\tilde{\nabla}^y_u V := \nabla^y_u V - \mathbf{L}_y(u,v), \tag{7.17}$$

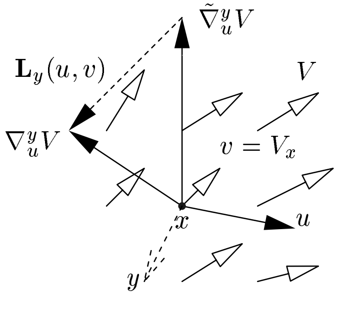
where $u, v \in T_xM$ and $V \in C^\infty(TM)$ with $V_x = v$. We call $\tilde{\nabla} = \{\tilde{\nabla}^y\}_{y \in TM\setminus\{0\}}$ the *Chern connection*.

In a standard local coordinate system (x^i, y^i) in TM, let

$$\tilde{\Gamma}^i_{jk}(y) := \Gamma^i_{jk}(y) - L^i_{jk}(y),$$

where $L^i_{jk} := g^{il}L_{jkl}$. We can express the Chern connection as follows

$$\tilde{\nabla}^y_u V = \Big\{ u(V^i)(x) + V^j(x)\tilde{\Gamma}^i_{jk}(y)u^k \Big\} \frac{\partial}{\partial x^i}|_x.$$



It follows from (7.16) and (7.17) that

$$\begin{aligned} \nabla^y_w Y &= \tilde{\nabla}^y_w Y, & (7.18)\\ \nabla^y_y V &= \tilde{\nabla}^y_y V, & (7.19)\end{aligned}$$

where $Y, V \in C^\infty(TM)$ and $w, y = Y_x \in T_xM$.

The Chern connection is characterized by the following equations

$$\tilde{\nabla}^Y_U V - \tilde{\nabla}^Y_V U = [U, V], \tag{7.20}$$

$$w[g_Y(U,V)] = g_Y(\tilde{\nabla}^Y_w U, V) + g_Y(U, \tilde{\nabla}^Y_w V) + \mathbf{C}_Y(U, V, \tilde{\nabla}^Y_w Y), \tag{7.21}$$

where $w \in T_xM$ and $U, V, Y \in C^\infty(TM)$. Clearly, (7.21) is simpler than (7.11).

The Chern connection also gives rise to a new quantity

$$P^i_{jkl}(y) := -\frac{\partial \tilde{\Gamma}^i_{jk}}{\partial y^l}(y) = -B^i_{jkl}(y) + \frac{\partial L^i_{jk}}{\partial y^l}(y). \tag{7.22}$$

Note that

$$P^i_{jkl}y^l = 0, \qquad P^i_{jkl}y^j = -L^i_{kl}.$$

For a vector $y \in T_xM \setminus \{0\}$, define $\mathbf{P}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by

$$\mathbf{P}_y(u, v, w) = P^i_{jkl}(y)u^j v^k w^l \ \frac{\partial}{\partial x^i}|_x. \tag{7.23}$$

We call $\mathbf{P} = \{\mathbf{P}_y\}_{y \in TM\setminus\{0\}}$ the *Chern curvature*. One can easily verify that

$$\mathbf{B} = 0 \quad \Longleftrightarrow \quad \mathbf{P} = 0.$$

Thus the Berwald metrics can also be characterized by $\mathbf{P} = 0$.

Connections are just tools. Since the Berwald connection is defined not only for Finsler metrics, but also for sprays, we will use the Berwald connection throughout this book.

7.3 Covariant Derivatives Along Geodesics

Once given a connection on a manifold, one can define the covariant derivatives of tensor fields along a curve.

Let $\mathbf{G}$ be a spray on an n-manifold and Γ^i_{jk} the Christoffel symbols of $\mathbf{G}$ in a standard local coordinates in TM. Let $c : [a, b] \to M$ be a C^∞ curve (possibly $\dot{c}(t) = 0$ for some t). A vector field $V = V(t)$ along c is a family of vectors

$$V(t) = V^i(t)\frac{\partial}{\partial x^i}\Big|_{c(t)},$$

where $V^i(t)$ are C^∞. Hence the tangent vector field of c,

$$\dot{c}(t) = \frac{dc^i}{dt}(t)\frac{\partial}{\partial x^i}\Big|_{c(t)},$$

is a vector field along c.

Fix a vector field $V = V(t) \neq 0$ along c. For a vector field $U(t)$ along c, we define

$$\nabla^V_{\dot{c}}U(t) := \Big\{\frac{dU^i}{dt}(t) + U^j(t)\Gamma^i_{jk}(V(t))\frac{dc^k}{dt}(t)\Big\}\frac{\partial}{\partial x^i}\Big|_{c(t)}. \tag{7.24}$$

At a point $c(t_o)$ where $\dot{c}(t_o) = 0$, (7.24) reduces to

$$\nabla^V_{\dot{c}}U(t_o) = \frac{dU^i}{dt}(t_o)\frac{\partial}{\partial x^i}\Big|_{c(t_o)}. \tag{7.25}$$

For simplicity, we always denote

$$\mathrm{D}_{\dot c}U(t) := \nabla_{\dot c}^{\dot c}U(t). \tag{7.26}$$

In local coordinates,

$$\mathrm{D}_{\dot c}U(t) = \Big\{ \frac{dU^i}{dt}(t) + U^j(t)N^i_j(\dot c(t)) \Big\} \frac{\partial}{\partial x^i}|_{c(t)}. \tag{7.27}$$

$\mathrm{D}_{\dot c}U(t)$ is called the *covariant derivative* of $U(t)$ along c.

Assume that c is a regular curve. Using $2G^i = N^i_j y^j$, we obtain

$$\mathrm{D}_{\dot c}\dot c = \Big\{ \ddot c^i + 2G^i(\dot c) \Big\} \frac{\partial}{\partial x^i}\Big|_{c(t)}. \tag{7.28}$$

Thus c is a geodesic if and only if

$$\mathrm{D}_{\dot c}\dot c = 0. \tag{7.29}$$

Equation (7.29) is just (4.6) in an index-free form.

Definition 7.3.1 A vector field $V = V(t)$ along c is said to be *parallel* if

$$\mathrm{D}_{\dot c}V = 0.$$

Let $p = c(a)$ and $q = c(b)$. We define a map $P_c : T_pM \to T_qM$ by

$$P_c(v) := V(b), \qquad v \in T_pM,$$

where $V(t)$ is a parallel vector field along c with $V(a) = v$. Clearly, P_c is a linear map. We call P_c the *parallel translation* along c.

Lemma 7.3.2 *Let (M, L) be a Finsler space. Assume that $c : [a, b] \to M$ is a geodesic from p to q. Then the parallel translation P_c preserves the inner products $g_{\dot c}$, i.e.,*

$$g_{\dot c(b)}\Big(P(u), P(v)\Big) = g_{\dot c(a)}(u, v), \qquad u, v \in T_pM. \tag{7.30}$$

Proof: Let $U = U(t)$, $V = V(t)$ be two parallel vector fields along c. By (7.11), we have

$$\frac{d}{dt}\Big(g_{\dot c}(U, V)\Big) = g_{\dot c(t)}\Big(\mathrm{D}_{\dot c}U, V\Big) + g_{\dot c(t)}\Big(U, \mathrm{D}_{\dot c}V\Big) = 0.$$

Thus $g_{\dot c}(U, V)$ is a constant. This implies (7.30) Q.E.D.

There is another way to define the covariant derivative of a vector field $U(t)$ along a curve c.

$$\tilde{\mathrm{D}}_{\dot c}U(t) := \nabla_{\dot c}^{U}U(t). \tag{7.31}$$

In local coordinates,

$$\tilde{\mathrm{D}}_{\dot c}U(t) = \Big\{ \frac{dU^i}{dt}(t) + \dot c^j(t) N^i_j(U(t)) \Big\} \frac{\partial}{\partial x^i}\Big|_{c(t)}. \tag{7.32}$$

A vector field $V = V(t) \neq 0$ along c is said to be *auto-parallel* if

$$\tilde{\mathrm{D}}_{\dot c} V(t) = 0.$$

Define a map $\tilde{P}_c : T_pM \to T_qM$ by

$$\tilde{P}_c(v) = V(b),$$

where $V = V(t)$ is a auto-parallel vector field along c with $V(a) = v$. It is easy to see that $\tilde{P}_c : T_pM \setminus \{0\} \to T_qM \setminus \{0\}$ is a diffeomorphism satisfying

$$\tilde{P}_c(\lambda v) = \lambda \tilde{P}_c(v), \qquad \lambda > 0$$

$\tilde{P}_c$ is not linear in general. We call $\tilde{P}_c$ the *auto-parallel translation* along c.

Lemma 7.3.3 *Let (M, L) be a Finsler space and $c : [a, b] \to M$ a geodesic from p to q. Then the auto-parallel translation $\tilde{P}_c : T_pM \to T_qM$ preserves the Minkowski functionals.*

Proof: It follows from (7.11) that for any auto-parallel vector field $V = V(t)$ along c,

$$\frac{d}{dt}\Big(g_V(V, V) \Big) = 2 g_V\Big(\tilde{\mathrm{D}}_{\dot c} V, V \Big) = 0.$$

Thus $F^2(V) = g_V(V, V)$ is a constant. This proves the lemma. Q.E.D.

By Lemmas 7.3.2 and 7.3.3, the reader can easily see that for Berwald metrics, $P_c = \tilde{P}_c$ is a linear transformation preserving the Minkowski functionals F_x in T_xM. This implies the following

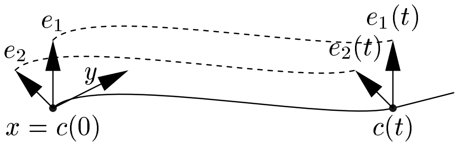
Proposition 7.3.4 ([Ic1]) On a positive definite Berwald space, Minkowski tangent spaces are linearly isometric to each other by parallel translations along geodesics.

Roughly speaking, on a positive definite Berwald space (M, F), the geometry of Minkowski tangent spaces does not change. Thus $\mathbf{S} = 0$. More precisely, we have

Proposition 7.3.5 ([Sh3]) *On any positive definite Berwald space (M, F), the S-curvature $\mathbf{S} = 0$ for the Busemann-Hausdorff volume form $d\mu_F$ and the Holmes-Thompson volume form $d\tilde{\mu}_F$.*

Proof: Let $0 \neq y \in T_xM$ be an arbitrary vector. Extend it to a geodesic field Y in a neighborhood of x. Let c be the geodesic with $\dot c(0) = y$ and $P_t := P_{c_t}$ denote the parallel translation along $c_t := c|_{[0,t]}$. Take an arbitrary basis $\{e_i\}_{i=1}^n$ for T_xM. We obtain a parallel frame along c,

$$e_i(t) := P_t(e_i), \qquad i = 1, \cdots, n.$$



Let μ_F denote the Busemann-Hausdorff measure of F which is defined in (5.1)-(5.3). Put

$$d\mu_F|_{c(t)} = \sigma_F(t)\omega^1(t) \wedge \cdots \wedge \omega^n(t),$$

where $\{\omega^i(t)\}$ be the basis for $T_{c(t)}M$ which are dual to $\{e_i(t)\}_{i=1}^n$.

$$\sigma_F(t) := \frac{\omega_n}{\mathrm{Vol}\Big\{(y^i),\ F(y^i e_i(t)) < 1\Big\}}.$$

According to Lemma 5.2.2, the S-curvature of F in the direction of $\dot{c}(t)$ is given by

$$\mathbf{S}(\dot{c}(t)) = \frac{d}{dt}\Big[\ln\Big(\frac{\sqrt{\det\big[g_{\dot{c}(t)}(e_i(t), e_j(t))\big]}}{\sigma_F(t)}\Big)\Big].$$

By Lemmas 7.3.2 and 7.3.3, we know that

$$g_{\dot{c}(t)}\Big(e_i(t), e_j(t)\Big) = g_y(e_i, e_j),$$

$$F\Big(y^i e_i(t)\Big) = F\Big(y^i e_i\Big).$$

Thus

$$\det\Big[g_{\dot{c}(t)}\Big(e_i(t), e_j(t)\Big)\Big] = \det\Big[g_y\Big(e_i, e_j\Big)\Big],$$

$$\mathrm{B}^n_{c(t)} = \Big\{(y^i),\ F\Big(y^i e_i(t)\Big) < 1\Big\} = \Big\{(y^i),\ F\Big(y^i e_i\Big) < 1\Big\} = \mathrm{B}^n_{c(0)}.$$

This implies $\mathbf{S}(\dot{c}(t)) = 0$. In particular, at $t = 0$, $\mathbf{S}(y) = 0$. By the same argument, we can also prove that the S-curvature vanishes for the Holmes-Thompson volume form. Q.E.D.

For a Finsler metric F on a manifold M, F_x induces a Riemannian metric $\hat{g}_x$ on $T_xM \setminus \{0\}$. See Example 7.1.1. We call $(T_xM \setminus \{0\}, \hat{g}_x)$ the *Riemannian tangent space* at x. If F is a Landsberg metric, then $\tilde{P}_c$ is a diffeomorphism preserving the induced Riemannian metrics $\hat{g}_x$ on $T_xM \setminus \{0\}$.

Proposition 7.3.6 ([Ic2]) *On a Landsberg space, Riemannian tangent spaces are isometric to each other by auto-parallel translations along geodesics.*

We omit the proof here. Refer to [Ic2] for details.

Take a geodesic $c(t)$ in a Finsler space (M, L) and a parallel vector field $V(t)$ along c. In local coordinates, $V(t) = V^i(t)\frac{\partial}{\partial x^i}|_{c(t)}$ satisfies the following system

$$\frac{dV^i}{dt} + V^j(t)N^i_j(\dot{c}) = 0. \tag{7.33}$$

We have

$$\frac{d}{dt}\Big[\mathbf{C}_{\dot{c}(t)}\Big(V(t), V(t), V(t)\Big)\Big] = \mathbf{L}_{\dot{c}(t)}\Big(V(t), V(t), V(t)\Big). \tag{7.34}$$

To prove (7.34), we simplify the above quantities and let

$$\begin{aligned} \mathbf{C}(t) : &= \mathbf{C}_{\dot{c}(t)}(V(t), V(t), V(t)), && (7.35)\\ \mathbf{L}(t) : &= \mathbf{L}_{\dot{c}(t)}(V(t), V(t), V(t)). && (7.36) \end{aligned}$$

Using (6.31) and (7.33), we obtain

$$\begin{aligned} \mathbf{C}'(t) &= \frac{d}{dt}\Big[C_{ijk}V^i(t)V^j(t)V^k(t)\Big] \\ &= \frac{dc^l}{dt}\frac{\partial C_{ijk}}{\partial x^l}V^iV^jV^k + \frac{d^2c^l}{dt^2}\frac{\partial C_{ijk}}{\partial y^l}V^iV^jV^k + 3C_{ijk}\frac{dV^i}{dt}V^jV^k \\ &= \Big\{\frac{dc^l}{dt}\frac{\partial C_{ijk}}{\partial x^l} - 2G^l\frac{\partial C_{ijk}}{\partial y^l} - 3C_{mjk}N^m_i\Big\}V^iV^jV^k \\ &= L_{ijk}V^iV^jV^k \\ &= \mathbf{L}(t). \end{aligned}$$

This proves (7.34). Thus on a Landsberg space ($\mathbf{L} = 0$), the Cartan torsion is constant along any geodesic. In other words, the Landsberg curvature measures the rate of changes of the Cartan torsion along geodesics. See [HaHoMa][Ic2][Ik][KawH] [Bw5] etc. for further discussion on Landsberg metrics.

Let $\{E_i(t)\}_{i=1}^n$ a parallel frame along c. Then

$$\frac{d}{dt}\Big[g_{\dot{c}(t)}\Big(E_i(t), E_j(t)\Big)\Big] = 0.$$

Thus $g_{ij} := g_{\dot{c}(t)}\Big(E_i(t), E_j(t)\Big)$ are constants. By definition, the mean Cartan torsion $\mathbf{I}$ is given by

$$\mathbf{I}_{\dot{c}(t)}\Big(U(t)\Big) = \sum_{ij=1}^n \mathbf{C}_{\dot{c}(t)}\Big(U(t), E_i(t), E_j(t)\Big)g^{ij}$$

and the mean Landsberg curvature $\mathbf{J}$ is given by

$$\mathbf{J}_{\dot c(t)}\Big(U(t)\Big) = \sum_{ij=1}^{n} \mathbf{L}_{\dot c(t)}\Big(U(t), E_i(t), E_j(t)\Big) g^{ij}.$$

It follows from (6.37) or (7.34) that

$$\frac{d}{dt}\Big[\mathbf{I}_{\dot c(t)}\Big(U(t)\Big)\Big] = \mathbf{J}_{\dot c(t)}\Big(U(t)\Big). \tag{7.37}$$

Thus $\mathbf{J}$ is the rate of changes of the mean Cartan torsion $\mathbf{I}$ along geodesics.

Example 7.3.1 Consider the Funk metric F on a strongly convex domain $\Omega \subset \mathrm{R}^n$. According to Example 6.2.2, the Landsberg curvature of F is given by

$$\mathbf{L} + \frac{1}{2} F\mathbf{C} = 0. \tag{7.38}$$

Let $c : (a, b) \to \Omega$ be an arbitrary unit speed geodesic of F. Define $\mathbf{C}(t)$ and $\mathbf{L}(t)$ as in (7.35) and (7.36) for a parallel vector field along c. It follows from (7.34) and (7.38) that

$$\mathbf{C}'(t) = -\frac{1}{2}\mathbf{C}(t). \tag{7.39}$$

Thus

$$\mathbf{C}(t) = e^{-\frac{1}{2}t}\mathbf{C}(0). \tag{7.40}$$

According to Section 4.3, F is positively complete. Assume that c is a unit speed geodesic of F defined on its maximal interval $(-\delta, \infty)$ with $c(0) = x$ and $y = \dot c(0) \in T_x\Omega$. Then

$$\delta = \ln\Big(1 + \frac{F(y)}{F(-y)}\Big).$$

Thus

$$\max|\mathbf{C}(t)| = \sqrt{1 + \frac{F(y)}{F(-y)}}|\mathbf{C}(0)|, \qquad \min|\mathbf{C}(t)| = 0. \tag{7.41}$$

We see that the Cartan torsion of F along any geodesic is bounded.

Now we consider the special case when the domain $\Omega = \mathrm{B}^n$ is the unit ball in R^n. In this case, F is given by

$$F := \frac{\sqrt{|y|^2 - \Big(|x|^2|y|^2 - \langle x, y\rangle^2\Big)}}{(1 - |x|^2)} + \frac{\langle x, y\rangle}{1 - |x|^2}, \tag{7.42}$$

Note that F Euclidean at the origin $x = 0$. Thus for any geodesic $c(t)$ passing through $c(0) = 0$, $\mathbf{C}(0) = 0$. Then by (7.41), we obtain that $\mathbf{C}(t) = 0$. This fact can be verified directly too, since F is explicitly given in (7.42). ♯

Chapter 8

Riemann Curvature

In 1854, B. Riemann introduced a notion of curvature tensor for Riemannian metrics. Seventy years later (1926), L. Berwald [Bw1][Bw8] successfully extended Riemann's curvature tensor to Finsler metrics and sprays. In this chapter, we will study the Riemann curvature of Riemannian metrics, Finsler metrics and sprays. Our approach is different from Riemann and Berwald.

8.1 Riemann Curvature of Sprays

The Riemann curvature is introduced via geodesics. Consider a spray on a manifold M

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}.$$

A geodesic $c(t)$ of $\mathbf{G}$ is characterized by

$$\ddot{c}^i + 2G^i(\dot{c}) = 0, \tag{8.1}$$

where $x(t) = (x^1(t), \cdots, x^n(t))$ denotes the coordinates of $c(t)$. For a vector field $U(t) = U^i(t)\frac{\partial}{\partial x^i}|_{c(t)}$ along c, its covariant derivative at $c(t)$ is given by

$$\mathrm{D}_{\dot{c}}U(t) = \Big\{ \frac{dU^i}{dt}(t) + U^j(t)N^i_j(\dot{c}(t)) \Big\} \frac{\partial}{\partial x^i}|_{c(t)},$$

where $N^i_j(y) := \frac{\partial G^i}{\partial y^j}(y)$ are the N-connection coefficients with $2G^i(y) = y^j N^i_j(y)$.

Consider a *geodesic variation* H of c, that is a map $H : (-\varepsilon, \varepsilon) \times [a, b] \to M$ such that the curves $c_u := H(u, \cdot)$ are geodesics with $c_0 = c$. Then the variation vector field along c

$$J(t) := \frac{\partial H}{\partial u}(0, t)$$

satisfies a 2nd order ordinary differential equation. More precisely, we have the following

Lemma 8.1.1 *There is a family of linear transformations* $\mathbf{R}_y : T_xM \to T_xM$, $y \in T_xM \setminus \{0\}$, *such that for any geodesic* $c(t)$ *and any geodesic variation* $H = H(u,t)$ *of* c, *the variation field* $J(t) := \frac{\partial H}{\partial u}(0,t)$ *satisfies the following Jacobi equation*

$$\mathrm{D}_{\dot c}\mathrm{D}_{\dot c}J(t) + \mathbf{R}_{\dot c(t)}(J(t)) = 0. \tag{8.2}$$

Proof: By assumption,

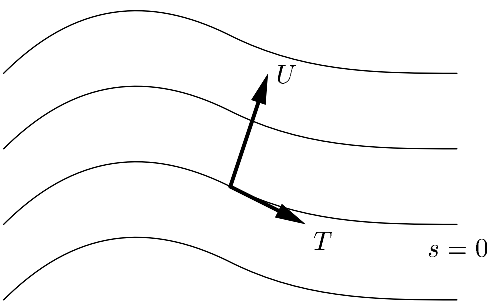
$$\frac{\partial^2 H^i}{\partial t^2} + 2G^i\Big(\frac{\partial H}{\partial t}\Big) = 0. \tag{8.3}$$

For simplicity, let

$$T = T^i\frac{\partial}{\partial x^i} := \frac{\partial H}{\partial t}, \qquad U = U^i\frac{\partial}{\partial x^i} := \frac{\partial H}{\partial u}.$$

Then (8.3) becomes

$$\frac{\partial T^i}{\partial t} + 2G^i(T) = 0. \tag{8.4}$$



Observe that

$$\frac{\partial}{\partial u}\Big[G^i(T)\Big] = U^k\frac{\partial G^i}{\partial x^k}(T) + \frac{\partial U^j}{\partial t}N^i_j(T), \tag{8.5}$$

$$\frac{\partial}{\partial t}\Big[N^i_j(T)\Big] = T^k\frac{\partial N^i_j}{\partial x^k}(T) + \frac{\partial T^k}{\partial t}\Gamma^i_{jk}(T) \tag{8.6}$$

$$= T^k\frac{\partial N^i_j}{\partial x^k}(T) - 2G^k(T)\Gamma^i_{jk}(T), \tag{8.7}$$

where $\Gamma^i_{jk} = \frac{\partial^2 G^i}{\partial y^j \partial y^k}$. Note that

$$\frac{\partial T^i}{\partial u} = \frac{\partial^2 H^i}{\partial u \partial t} = \frac{\partial U^i}{\partial t}. \tag{8.8}$$

Differentiating (8.4) with respect to u yields

$$\frac{\partial^2 U^i}{\partial t^2} = -2U^k\frac{\partial G^i}{\partial x^k}(T) - 2\frac{\partial U^j}{\partial t}N^i_j(T). \tag{8.9}$$

Using the above identities, one obtains

$$\begin{aligned} \mathrm{D}_T\mathrm{D}_T U &= \mathrm{D}_T\Big[\Big(\frac{\partial U^i}{\partial t} + U^j N^i_j(T)\Big)\frac{\partial}{\partial x^i}\Big] \\ &= -U^k\Big\{2\frac{\partial G^i}{\partial x^k} - T^j\frac{\partial N^i_k}{\partial x^j} + 2G^j\Gamma^i_{jk} - N^i_j N^j_k\Big\}\Big|_T \frac{\partial}{\partial x^i}. \end{aligned} \tag{8.10}$$

For a vector $y \in T_xM \setminus \{0\}$, let

$$R^i_{\ k}(y) := 2\frac{\partial G^i}{\partial x^k}(y) - y^j\frac{\partial N^i_k}{\partial x^j}(y) + 2G^j(y)\Gamma^i_{jk}(y) - N^i_j(y)N^j_k(y). \tag{8.11}$$

Define $\mathbf{R}_y : T_xM \to T_xM$ by

$$\mathbf{R}_y(v) := R^i_{\ k}(y)v^k\frac{\partial}{\partial x^i}|_x, \qquad v = v^k\frac{\partial}{\partial x^k}|_x. \tag{8.12}$$

$\mathbf{R}_y$ is a well-defined linear transformation satisfying $\mathbf{R}_y(y) = 0$. We can write (8.10) as follows

$$\mathrm{D}_T\mathrm{D}_T U + \mathbf{R}_T(U) = 0. \tag{8.13}$$

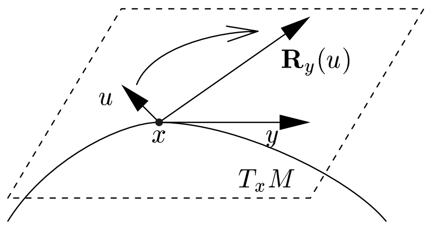
The equation (8.13) restricted to c is just (8.2). This completes the proof. Q.E.D.

Rewriting (8.11) as follows

$$\begin{aligned} R^i_{\ k} &= 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j\partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^j\partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k} & (8.14)\\ &= \Big\{\frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^i_{ks}\Gamma^s_{jl} - \Gamma^s_{jk}\Gamma^i_{ls}\Big\}y^j y^l, & (8.15) \end{aligned}$$

Definition 8.1.2 The quantity $\mathbf{R}$ defined in (8.12) is called the *Riemann curvature.*

The Riemann curvature was originally introduced by B. Riemann in 1854 for (sprays induced by) Riemannian metrics. P. Finsler first studied the variational problem on Finsler metrics. But he did not introduce any notion of curvature for Finsler metrics. It was L. Berwald [Bw1] who first successfully extended it to Finsler metrics, that can be expressed in terms of the Finsler spray coefficients as in (8.14). Thus, the Riemann curvature can be naturally defined for sprays without any modification [Dg1] [Dg2] [Bw8]. About the same time, D. Kosambi introduced a quantity for semisprays via the second variation of semigeodesics [Ko2][Ko3]. It turns out that Kosambi's quantity is exactly the Riemann curvature when the semispray is a spray. Unfortunately, many names have been given to this distinguished quantity in various settings (index-dependent or index-free). Some people call it the Jacobi endomorphism and the h-curvature, etc. In this book , we still call it the *Riemann curvature.*



The Riemann curvature can also be viewed as a section of $\mathcal{H}^*TM \otimes \mathcal{V}TM$ if we express it in the following form

$$R^i_k(y)dx^k \otimes \frac{\delta}{\delta x^i}.$$

See (4.18) and (4.9) for the definition of $\mathcal{H}^*TM$ and $\mathcal{V}TM$ respectively. Put

$$R^i{}_{kl} := \frac{1}{3}\Big\{\frac{\partial R^i_k}{\partial y^l} - \frac{\partial R^i_l}{\partial y^k}\Big\}, \tag{8.16}$$

$$R_j{}^i{}_{kl} := \frac{1}{3}\Big\{\frac{\partial^2 R^i_k}{\partial y^j \partial y^l} - \frac{\partial^2 R^i_l}{\partial y^j \partial y^k}\Big\}. \tag{8.17}$$

We obtain two more tensors on $TM \setminus \{0\}$.

$$R^i{}_{kl}dx^k \otimes dx^l \otimes \frac{\delta}{\delta x^i}, \qquad R_j{}^i{}_{kl}dx^j \otimes dx^k \otimes dx^l \otimes \frac{\delta}{\delta x^i}.$$

It follows from (8.17) that

$$R_j{}^i{}_{kl} + R_j{}^i{}_{lk} = 0, \tag{8.18}$$

$$R_j{}^i{}_{kl} + R_k{}^i{}_{lj} + R_l{}^i{}_{jk} = 0. \tag{8.19}$$

The local quantities $R^i_k(y), R^i{}_{kl}(y)$ and $R_j{}^i{}_{kl}(y)$ all live on $TM \setminus \{0\}$. They are actually the coefficients of tensors on $TM \setminus \{0\}$. See Chapter 9 below.

A direct computation using (8.17) yields

$$R_j{}^i{}_{kl} = \frac{\delta \Gamma^i_{jl}}{\delta x^k} - \frac{\delta \Gamma^i_{jk}}{\delta x^l} + \Gamma^i_{ks}\Gamma^s_{jl} - \Gamma^s_{jk}\Gamma^i_{ls}. \tag{8.20}$$

Thus

$$R^i{}_{kl} = R_j{}^i{}_{kl}y^j, \qquad R^i_k = R_j{}^i{}_{kl}y^jy^l. \tag{8.21}$$

Remark 8.1.3 Every spray determines a unique horizontal distribution $\mathcal{H}TM = \mathrm{span}\Big\{\frac{\delta}{\delta x^i}\Big\}$ on $TM\setminus\{0\}$ (see (4.14)). Observe that

$$\Big[\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\Big] = \Big(\frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}\Big)\frac{\partial}{\partial y^i} = -R^i{}_{kl}\frac{\partial}{\partial y^i}.$$

Thus $\mathbf{R} = 0$ if and only if $\mathcal{H}TM$ is integrable, namely, at every point $y \in TM\setminus\{0\}$, there is an n-dimensional submanifold N of TM passing through y such that the tangent space of N at every point $y' \in N$ coincides with $\mathcal{H}_{y'}TM$. This observation was made by X. Mo for Finsler metrics [Mo].

Assume that a spray $\mathbf{G}$ is affine. Then the Berwald connection ∇ of $\mathbf{G}$ is affine. The Berwald connection coincides with the N-connection D. Since the Christoffel symbols Γ^i_{jk} are functions of x only, (8.20) simplifies to

$$R_j{}^i{}_{kl}(x) = \frac{\partial\Gamma^i_{jl}}{\partial x^k}(x) - \frac{\partial\Gamma^i_{jk}}{\partial x^l} + \Gamma^i_{ks}(x)\Gamma^s_{jl}(x) - \Gamma^s_{jk}(x)\Gamma^i_{ls}(x). \tag{8.22}$$

Note that $R_j{}^i{}_{kl}(x)$ live on M. We obtain a tensor R on M

$$\mathrm{R}(u,v)w := R_j{}^i{}_{kl}(x)u^kv^lw^j\frac{\partial}{\partial x^i}|_x, \tag{8.23}$$

where $u = u^k\frac{\partial}{\partial x^k}|_x, v = v^l\frac{\partial}{\partial x^l}|_x$ and $w = w^j\frac{\partial}{\partial x^j}|_x$. We call R the *Riemann curvature tensor* of D. The Riemann curvature tensor can be expressed in an index-free form.

$$\mathrm{R}(U,V)W = \Big(\mathrm{D}_U\mathrm{D}_V - \mathrm{D}_V\mathrm{D}_U - \mathrm{D}_{[U,V]}\Big)W,$$

where $U = U^k\frac{\partial}{\partial x^k}, V = V^l\frac{\partial}{\partial x^l}$ and $W = W^j\frac{\partial}{\partial x^j}$ are vector fields on M,

Now we derive a useful formula for the Riemann curvature. Let $\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}$ be a spray on a manifold M and $\mathbf{Q} = Q^i(y)\frac{\partial}{\partial y^i}$ a vertical vector field on $TM\setminus\{0\}$ with

$$Q^i(\lambda y) = \lambda^2Q^i(y), \qquad \lambda > 0.$$

Let

$$\omega^i = dx^i, \qquad \omega^{n+i} = \delta y^i := dy^i + N^i_j(y)dx^j.$$

We obtain a natural co-frame $\{\omega^i, \omega^{n+i}\}_{i=1}^n$ on $TM\setminus\{0\}$. Let

$$\omega_j{}^i := \Gamma^i_{jk}dx^k.$$

For any vertical vector field $X = X^i\frac{\partial}{\partial y^i}$ on TM, the following map from tangent vectors to vertical tangent vectors on $TM\setminus\{0\}$

$$\nabla X := \Big\{dX^i + X^j\omega_j{}^i\Big\}\otimes\frac{\partial}{\partial y^i} \tag{8.24}$$

is independent of standard local coordinate system (x^i, y^i) in TM. We call ∇ the *Berwald connection* and ${\omega_j}^i$ the *Berwald connection forms*. Define $Q^i_{\cdot k}$, $Q^i_{;k}$, $Q^i_{\cdot k;l}$ and $Q^i_{;k\cdot l}$ by

$$dQ^i + Q^j {\omega_j}^i = Q^i_{;k}\omega^k + Q^i_{\cdot k}\omega^{n+k}. \tag{8.25}$$

$$dQ^i_{\cdot k} + Q^j_{\cdot k}{\omega_j}^i - Q^i_{\cdot j}{\omega_k}^j = Q^i_{\cdot k;l}\omega^l + Q^i_{\cdot k\cdot l}\omega^{n+l}. \tag{8.26}$$

$Q^i_{\cdot k}$ and $Q^i_{;k}$, etc. are called the coefficients of the covariant derivatives of $\mathbf{Q} = Q^i \frac{\partial}{\partial y^i}$ with respect to the natural basis $\{\frac{\partial}{\partial y^i}\}$ for $\mathcal{V}TM$. See Section 9.1 for more details. Based on (8.14), we introduce a new quantity

$$\mathbf{H}_y(u) := H^i_k(y)u^k \frac{\partial}{\partial x^i}|_x, \qquad u = u^i \frac{\partial}{\partial x^i}|_x \in T_xM, \tag{8.27}$$

where

$$H^i_k := 2Q^i_{;k} - Q^i_{\cdot k;j}y^j + 2Q^jQ^i_{\cdot j\cdot k} - Q^i_{\cdot j}Q^j_{\cdot k}. \tag{8.28}$$

By (8.14), we immediately obtain the following

Lemma 8.1.4 *The Riemann curvature of* $\tilde{\mathbf{G}} := \mathbf{G} - 2\mathbf{Q}$ *is related to that of* $\mathbf{G}$ *by*

$$\tilde{\mathbf{R}} = \mathbf{R} + \mathbf{H}. \tag{8.29}$$

Take an arbitrary local frame $\{e_i\}$ for TM. Let $\{e^v_i\}$ denote the vertical lift of $\{e_i\}$ for $\mathcal{V}TM$. If $e_i = a^j_i \frac{\partial}{\partial x^i}|_x$, then $e^v_i := a^j_i \frac{\partial}{\partial y^j}|_y$, where $y \in T_xM \setminus \{0\}$. By the canonical bundle isomorphism $\mathcal{H}TM \to \mathcal{V}TM$, we obtain a horizontal frame $\{e^h_i\}$ for $\mathcal{H}TM$. If $e_i := a^j_i \frac{\partial}{\partial x^i}|_x$, then $e^h_i := a^j_i \frac{\delta}{\delta x^j}|_y$, where $y \in T_xM \setminus \{0\}$. Let $\{\omega^i, \omega^{n+i}\}$ denote the co-frame dual to $\{e^h_i, e^v_i\}$. The Berwald connection ∇ determines a set of connection forms ${\omega_j}^i$ by

$$\nabla e^v_j = {\omega_j}^i \otimes e^v_i.$$

Express $\mathbf{Q} = Q^i e^v_i$. One can define the covariant derivatives with respect to $\{\omega^i, \omega^{n+i}\}$ using ${\omega_j}^i$. $\mathbf{H}_y(u) = H^i_k(y)u^k e_i$ can be expressed in terms of $Q^i_{;k}$ and $Q^i_{\cdot k}$ etc. by the same formula (8.28), where $Q^i_{;k}$ and $Q^i_{\cdot k}$, etc. are the coefficients of the covariant derivatives of $\mathbf{Q} = Q^i e^v_i$ with respect to $\{e^v_i\}$. Of course, (8.29) still holds. The formula (8.29) is useful in computing the Riemann curvature of certain Finsler metrics, such as Randers metrics.

Definition 8.1.5 A spray on a manifold M is said to be *isotropic* if the Riemann curvature is in the following form

$$\mathbf{R}_y(u) = R(y)\, u + \tau_y(u)\, y, \qquad u \in T_xM, \tag{8.30}$$

where $\tau_y \in T^*_xM$ with $\tau_y(y) = -R(y)$. It is said to be *R-flat* if $\mathbf{R} = 0$.

A spray is said to be *flat* if at every point, there is a local coordinate system in which

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i}.$$

Clearly, flat sprays must be R-flat. The converse is not true. Can you find a R-flat spray which is not flat? The following proposition is due to L. Berwald and J. Douglas.

Proposition 8.1.6 *A spray is flat if and only if* $\mathbf{B} = 0$ *and* $\mathbf{R} = 0$.

Proof: We shall only sketch the proof here. First, $\mathbf{B} = 0$ implies that the Berwald connection ∇ is affine. In this case, the Berwald connection coincides with the N-connection D. From the definition of $R_j{}^i{}_{kl}$, $\mathbf{R} = 0$ implies that the Riemann curvature tensor R of D vanishes. A classical theorem on linear connections implies that there is a local coordinate system in which $\Gamma^i_{jk} = 0$. Thus $\mathbf{G}$ is flat. See [BaChSh1] for details. Q.E.D.

Definition 8.1.7 The trace of the Riemann curvature

$$\mathbf{Ric}(y) := (n-1)R(y) = R^m_m(y) \tag{8.31}$$

is called the *Ricci curvature* and $R(y) = \frac{1}{n-1}\mathbf{Ric}(y)$ is called the *Ricci-scalar*.

Example 8.1.1 Consider the following spray on R^2

$$\mathbf{G} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + 2u\varphi_v\frac{\partial}{\partial u} - 2\Big(\varphi - v\varphi_v\Big)\frac{\partial}{\partial v},$$

where $\varphi = \varphi(x, y, u, v)$ is a C^∞ function on $\mathrm{R}^2 \times (\mathrm{R}^2 \setminus \{0\})$ satisfying

$$\varphi(x, y, \lambda u, \lambda v) = \lambda^2\varphi(x, y, u, v), \qquad \lambda > 0.$$

A direct computation yields

$$\mathbf{Ric} = 2\varphi_y(x, y, u, v).$$

Thus $\mathbf{Ric} = 0$ if and only if φ is independent of y. ♯

Definition 8.1.8 A spray $\mathbf{G}$ is said to be *Ricci-constant* if the Ricci scalar R satisfies

$$R_{;i} := R_{x^i} - N^j_i R_{y^j} = 0. \tag{8.32}$$

It is said to be *weakly Ricci-constant* if

$$y^i R_{;i} = y^i R_{x^i} - 2G^i R_{y^i} = 0. \tag{8.33}$$

It is said to be *Ricci-flat* if $R = 0$.

Lemma 8.1.9 *Suppose that a spray* $\mathbf{G}$ *is weakly Ricci-constant. Then for any geodesic c,*

$$\mathbf{Ric}(\dot c(t)) = constant.$$

Proof: Observe that

$$\begin{aligned} \frac{d}{dt}\Big[R\big(\dot c(t)\big)\Big] &= \dot c^i R_{x^i}\big(\dot c(t)\big) + \ddot c^i R_{y^i}\big(\dot c(t)\big) \\ &= \dot c^i R_{x^i}\big(\dot c(t)\big) - 2G^i\big(\dot c(t)\big) R_{y^i}\big(\dot c(t)\big) = 0. \end{aligned}$$

This proves the lemma. Q.E.D.

Note that $\mathbf{R} = 0$ implies that $\mathbf{Ric} = 0$. The converse is not true, even in dimension two. See Example 8.1.2 below.

Example 8.1.2 Consider the following affine spray on R^3

$$\mathbf{G} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} - 2au^2\frac{\partial}{\partial u} - 2bv^2\frac{\partial}{\partial v} - 2cw^2\frac{\partial}{\partial w}, \tag{8.34}$$

where $a = a(x,y,z)$, $b = b(x,y,z)$ and $c = c(x,y,z)$. A direct computation yields

$$R^1_1 = -2a_y\, uv - 2a_z\, uw, \qquad R^1_2 = 2a_y\, u^2, \qquad R^1_3 = 2a_z\, u^2$$

$$R^2_1 = 2b_x\, v^2, \qquad R^2_2 = -2b_x\, uv - 2b_z\, vw, \qquad R^2_3 = 2b_z\, v^2$$

$$R^3_1 = 2c_x\, w^2, \qquad R^3_2 = 2c_y\, w^2, \qquad R^3_3 = -2c_x\, uw - 2c_y\, vw.$$

Thus $\mathbf{R} = 0$ if and only if

$$a = a(x), \qquad b = b(y), \qquad c = c(z).$$

The Ricci curvature is given by

$$\mathbf{Ric} = -4\Big[\Big(a_y + b_x\Big)uv + \Big(a_z + c_x\Big)uw + \Big(b_z + c_y\Big)vw\Big].$$

Thus $\mathbf{Ric} = 0$ if and only if

$$a_y + b_x = 0, \qquad a_z + c_x = 0, \qquad b_z + c_y = 0.$$

There are $a = a(x,y,z), b = b(x,y,z)$ and $c = c(x,y,z)$ such that the corresponding spray $\mathbf{G}$ is Ricci-flat, but not R-flat. ♯

By definition, the Ricci curvature is the trace of the Riemann curvature. For a two-dimensional spray, the Riemann curvature can be expressed in a special form.

Lemma 8.1.10 *On a surface M, the Riemann curvature of a spray must be in the following form*

$$\mathbf{R}_{\mathbf{y}} = R(\mathbf{y})\, I + \tau_{\mathbf{y}}\, \mathbf{y}, \tag{8.35}$$

where $R = \mathbf{Ric}$ denotes the Ricci scalar, $I : T_xM \to T_xM$ denotes the identity map and $\tau_{\mathbf{y}} \in T_x^*M$ is a linear functional satisfying

$$\tau_{\mathbf{y}}(\mathbf{y}) = -R(\mathbf{y}). \tag{8.36}$$

Proof: To prove (8.35), we just need the fact $\mathbf{R}_{\mathbf{y}}(\mathbf{y}) = 0$. Take an arbitrary basis $\{e_1, e_2\}$ for T_xM and let $\mathbf{y} = ue_1 + ve_2$. Observe that $R = R^1_1 + R^2_2$ and

$$R^2_1 u - (R^1_1 - R)v = R^2_1 u + R^2_2 v = 0,$$
$$R^1_2 v - (R^2_2 - R)u = R^1_1 u + R^1_2 v = 0.$$

Let

$$\tau_1 : \;=\; \frac{R^2_1}{v} = \frac{R^1_1 - R}{u}, \tag{8.37}$$

$$\tau_2 : \;=\; \frac{R^1_2}{u} = \frac{R^2_2 - R}{v}. \tag{8.38}$$

It follows from (8.37) and (8.38) that for a vector $\mathbf{u} = \eta e_1 + \zeta e_2 \in T_xM$,

$$R^1_1 \eta + R^1_2 \zeta \;=\; R\eta + \Big(\tau_1 \eta + \tau_2 \zeta\Big) u, \tag{8.39}$$

$$R^2_1 \eta + R^2_2 \zeta \;=\; R\zeta + \Big(\tau_1 \eta + \tau_2 \zeta\Big) v. \tag{8.40}$$

These two identities are just (8.35). Q.E.D.

Now we compute the Riemann curvature of a spray $\mathbf{G}$ on a surface M. In a standard local coordinate system (x, y, u, v) in TM, express $\mathbf{G}$ by

$$\mathbf{G} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - 2G(x,y,u,v)\frac{\partial}{\partial u} - 2H(x,y,u,v)\frac{\partial}{\partial v}. \tag{8.41}$$

Using (8.14), we obtain

$$\begin{aligned}
R^1_1 &= (G_{xv} - G_{yu})v + 2GG_{uu} + 2HG_{uv} - G_uG_u - G_vH_u \\
R^1_2 &= -(G_{xv} - G_{yu})u + 2GG_{uv} + 2HG_{vv} - G_uG_v - G_vH_v \\
R^2_1 &= -(H_{yu} - H_{xv})v + 2GH_{uu} + 2HH_{uv} - H_uG_u - H_uH_v \\
R^2_2 &= (H_{yu} - H_{xv})u + 2GH_{uv} + 2HH_{vv} - H_uG_v - H_vH_v.
\end{aligned}$$

The Ricci curvature $\mathbf{Ric} = R = R^1_1 + R^2_2$ is given by

$$\begin{aligned}
R \;=\; & 2G_x + 2H_y - G_u^2 - H_v^2 - 2H_uG_v - u(G_u + H_v)_x \\
& -v(G_u + H_v)_y + 2G(G_u + H_v)_u + 2H(G_u + H_v)_v.
\end{aligned} \tag{8.42}$$

Let

$$P := \frac{2}{u}G - (G_u + H_v) = \frac{v}{u}G_v - H_v. \tag{8.43}$$

R can be expressed by

$$R = -2\frac{v}{u}G_y + 2H_y + P_x u + P_y v - 2GP_u - 2HP_v - P^2. \tag{8.44}$$

If $P = 0$, then

$$R = -2\frac{v}{u}G_y + 2H_y. \tag{8.45}$$

Now we express the spray coefficients G and H in the following form

$$\begin{aligned} G &= \frac{1}{2}\Theta\Big(x, y, \frac{v}{u}\Big)u^2 \\ H &= \frac{1}{2}\Theta\Big(x, y, \frac{v}{u}\Big)uv - \frac{1}{2}\Phi\Big(x, y, \frac{v}{u}\Big)u^2, \end{aligned}$$

where $\Theta = \Theta(x, y, \xi)$ and $\Phi = \Phi(x, y, \xi)$ are functions of (x, y, ξ). The function P in (8.43) simplifies to

$$P = \frac{u}{2}\Big(\Phi_\xi - \Theta\Big).$$

It follows from (8.44) that R in the direction $(u, v) = (1, \xi)$ is given by

$$\begin{aligned} R &= -\Phi_y + \frac{1}{2}(\Phi_\xi - \Theta)_y\xi + \frac{1}{2}(\Phi_\xi - \Theta)_x \\ &\quad + \frac{1}{2}\Phi(\Phi_\xi - \Theta)_\xi - \frac{1}{4}(\Phi_\xi + \Theta)(\Phi_\xi - \Theta). \end{aligned} \tag{8.46}$$

When $P = 0$, i.e., $\Theta = \Phi_\xi$, the Riemann curvature in the direction $(u, v) = (1, \xi)$ is given by

$$\begin{aligned} R^1_1 &= \frac{1}{2}\Big(\Phi_{x\xi\xi} - 2\Phi_{y\xi} + \Phi_{y\xi\xi}\xi\Big)\xi + \frac{1}{2}\Phi\Phi_{\xi\xi\xi}\xi, \\ R^2_2 &= -\frac{1}{2}\Big(\Phi_{x\xi\xi} - 2\Phi_{y\xi} + \Phi_{y\xi\xi}\xi\Big)\xi - \frac{1}{2}\Phi\Phi_{\xi\xi\xi}\xi - \Phi_y. \end{aligned}$$

R^1_2 and R^2_1 are given by $R^1_2 = -R^1_1/\xi$ and $R^2_1 = -R^2_2\xi$. The Ricci curvature in the direction of $(u, v) = (1, \xi)$ is given by

$$R = -\Phi_y. \tag{8.47}$$

By the above formulas, we immediately obtain the following

Proposition 8.1.11 *Let* $\mathbf{G} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - 2G\frac{\partial}{\partial u} - 2H\frac{\partial}{\partial v}$ *be a two-dimensional spray, where* G *and* H *are in the form*

$$G = \frac{1}{2}\Phi_\xi\Big(x, y, \frac{v}{u}\Big)u^2 \tag{8.48}$$

$$H = \frac{1}{2}\Phi_\xi\Big(x, y, \frac{v}{u}\Big)uv - \frac{1}{2}\Phi\Big(x, y, \frac{v}{u}\Big)u^2, \tag{8.49}$$

where $\Phi = \Phi(x, y, \xi)$ is an arbitrary function. The Riemann curvature vanishes if and only if

$$\Phi_y = 0, \qquad \Phi_{x\xi\xi} + \Phi\Phi_{\xi\xi\xi} = 0. \tag{8.50}$$

Let $k \neq 0$ be an arbitrary constant and $f(s)$ a solution to the following ODE

$$f(s)f'''(s) = kf''(s). \tag{8.51}$$

Let

$$\Phi := \frac{1}{kx} f(\xi). \tag{8.52}$$

It is easy to verify that Φ satisfies (8.50). Thus the corresponding spray $\mathbf{G}$ defined in (8.48) and (8.49) has vanishing Riemann curvature. This spray is not projectively affine, because no solution of (8.51) is a polynomial of degree three or less. See Corollary 13.2.5.

Example 8.1.3 Let $\mathbb{B}^2$ be the standard unit ball in $\mathbb{R}^2$. Let

$$\varphi(x, y, u, v) := \frac{\sqrt{u^2 + v^2 - (xv - yu)^2} + (xu + yv)}{1 - x^2 - y^2}.$$

φ is a function on $T\mathbb{B}^2 = \mathbb{B}^2 \times R^2$. Let $\mathbf{G}$ be a two-dimensional spray on $\mathbb{B}^2$,

$$\mathbf{G} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - 2\varphi u\frac{\partial}{\partial u} - 2\varphi v\frac{\partial}{\partial v}.$$

It is easy to verify that $\mathbf{R} = 0$. ♯

8.2 Riemann Curvature of Finsler Metrics

Every Finsler metric induces a spray by (4.29) and (4.30). Thus the Riemann curvature of the Finsler metric is defined to be that of the induced spray. Let us take a look at the following example before we discuss some special properties of the Riemann curvature for Finsler metrics.

Example 8.2.1 Consider a Randers metric $F = \alpha + \beta$ on an n-dimensional manifold M, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form satisfying

$$\|\beta\|_x := \sqrt{a^{ij}(x)b_i(x)b_j(x)} < 1.$$

It is natural to express the Riemann curvature of F in terms of that of α and the covariant derivatives of β. In general, it is very complicated. A complete solution is given in [YaSh][ShKi][Ma6], etc. When β is a close 1-form, the formula for the Riemann curvature is much simpler.

Assume that $\beta = b_i(x)dx^i$ is close. Let $b_{i;j}$ and $b_{i;j;k}$ denote the components of the covariant derivatives of β with respect to α. Put

$$\Phi := b_{i;j}(x)y^iy^j, \qquad \Psi := b_{i;j;k}(x)y^iy^jy^k. \tag{8.53}$$

Let G^i and $\bar{G}^i$ denote the spray coefficients of F and α respectively. According to Example 5.2.2,

$$G^i = \bar{G}^i + \frac{\Phi}{2F}y^i.$$

Plugging them into (8.14) yields

$$R^i_k = \bar{R}^i_k + \Big[\frac{3}{4}\Big(\frac{\Phi}{F}\Big)^2 - \frac{\Psi}{2F}\Big]h^i_k + \eta_k y^i, \tag{8.54}$$

where[1]

$$h^i_k := \delta^i_k - F^{-2}(y)g_{kl}(y)y^ly^i, \qquad \eta_k = -F^{-1}(b_{i;j;k} - b_{i;k;j})y^iy^j = -F^{-1}b_j\bar{R}^j{}_k.$$

In a index-free form,

$$\mathbf{R}_y = \bar{\mathbf{R}}_y + \Big[\frac{3}{4}\Big(\frac{\Phi}{F}\Big)^2 - \frac{\Psi}{2F}\Big]\Big(I - F^{-2}(y)g_y(y,\cdot)y\Big) - F^{-1}(y)\beta(\bar{\mathbf{R}}_y). \tag{8.55}$$

It follows from (8.55) that

$$\mathbf{Ric} = \overline{\mathbf{Ric}} + (n-1)\Big[\frac{3}{4}\Big(\frac{\Phi}{F}\Big)^2 - \frac{\Psi}{2F}\Big]. \tag{8.56}$$

Without assuming that β be close, the formulas for $\mathbf{R}$ and $\mathbf{Ric}$ are much more complicated. See [YaSh][Ma6][Ma8]. We can compute them using (5.28) and (8.29). A good example is given in [BaSh2] where we compute the Riemann curvature for a special family of Randers metric on S^3.

♯

An important fact on Finsler sprays is that the Riemann curvature $\mathbf{R}_y$ is self-adjoint with respect to g_y. More precisely, we have the following

Proposition 8.2.1 *Let (M, L) be a Finsler space. For any $y \in T_xM \setminus \{0\}$, satisfies*

$$g_y\Big(\mathbf{R}_y(u), v\Big) = g_y\Big(u, \mathbf{R}_y(v)\Big). \tag{8.57}$$

To prove (8.57), one just needs to verify the following

$$g_{il}R^l_k = g_{kl}R^l_i. \tag{8.58}$$

The detailed argument is given in Remark 8.4.4 and Section 10.1 below.

From Proposition 8.2.1, we see that the Riemann curvature $\mathbf{R}_y : T_xM \to T_xM$ is self-adjoint with respect to g_y for every $y \in T_xM \setminus \{0\}$. This implies

[1] In the published book, the last term $\eta_k y^i$ is missing.

that $\mathbf{R}_y$ is diagonalizable as a linear transformation. There are sprays whose Riemann curvature $\mathbf{R}_y$ is not diagonalizable. See Example 8.2.4 below. Thus not all sprays can be induced by a positive definite Finsler metric.

Now let us take a look at Finsler metrics whose sprays are isotropic.

Lemma 8.2.2 *Let L be a Finsler metric on a manifold M. Suppose that the induced spray is isotropic, then there exists a scalar function $\lambda(y)$ on $TM \setminus \{0\}$ such that Riemann curvature is in the form*

$$\mathbf{R}_y(u) = \lambda(y)\Big\{g_y(y,y)u - g_y(y,u)y\Big\}, \qquad u \in T_xM. \tag{8.59}$$

Proof: By assumption, the Riemann curvature is in the following form

$$\mathbf{R}_y = R(y)\, I + \tau_y\, y, \quad y \in T_xM \setminus \{0\}, \tag{8.60}$$

where $\tau_y \in T_x^*M$ with $\tau_y(y) = -R(y)$. It follows from (8.60) that for arbitrary $u, v \in T_xM$,

$$\begin{aligned} g_y(\mathbf{R}_y(u), v) &= R(y)g_y(u,v) + \tau_y(u)g_y(y,v), \\ g_y(\mathbf{R}_y(v), u) &= R(y)g_y(v,u) + \tau_y(v)g_y(y,u). \end{aligned}$$

By Proposition 8.2.1, we obtain

$$\tau_y(u)g_y(y,v) = \tau_y(v)g_y(y,u).$$

Taking $v = y$ yields

$$\tau_y(u) = \frac{\tau_y(y)}{g_y(y,y)}g_y(y,u) = -\frac{R(y)}{L(y)}g_y(y,u).$$

Letting $\mathbf{K}(y) := R(y)/L(y)$, we obtain (8.59). Q.E.D.

Lemma 8.2.2 leads to the following

Definition 8.2.3 ([Bw3][Bw7]) A Finsler metric L is said to be *of scalar curvature* if there is a scalar function $\mathbf{K}(y)$ on $TM \setminus \{0\}$ such that the Riemann curvature is in the following form

$$\mathbf{R}_y(u) = \lambda(y)\Big\{g_y(y,y)u - g_y(y,u)y\Big\}, \qquad u \in T_xM. \tag{8.61}$$

The function $\lambda(y)$ is called the *curvature scalar* . L is of *constant curvature* λ, if $\lambda(y) = \lambda$. L is of zero curvature if $\lambda = 0$.

Let L be a Finsler metric on a surface M. By Lemma 8.1.10, we know that the Riemann curvature of L is in the form (8.35). By Lemma 8.2.2, we conclude that L is of scalar curvature with curvature scalar $\mathbf{K}(\mathbf{y})$ and the Riemann curvature is given by

$$\mathbf{R}_{\mathbf{y}}(\mathbf{u}) = \mathbf{K}(\mathbf{y})\Big\{L(\mathbf{y})\;\mathbf{u} - g_{\mathbf{y}}(\mathbf{y},\mathbf{u})\;\mathbf{y}\Big\}. \tag{8.62}$$

We come to the same conclusion that every two-dimensional Finsler metric is of scalar curvature. Note that the Ricci curvature $\mathbf{Ric}(\mathbf{y})$ is related to $\mathbf{K}(\mathbf{y})$ by

$$\mathbf{K}(\mathbf{y}) = \frac{\mathbf{Ric}(\mathbf{y})}{L(\mathbf{y})}. \tag{8.63}$$

The scalar function $\mathbf{K}(\mathbf{y})$ on $TM \setminus \{0\}$ is the Gauss curvature. From (8.62), we see that $\mathbf{R} = 0$ if and only if $\mathbf{Ric} = 0$. Thus any two-dimensional spray with $\mathbf{Ric} = 0$ and $\mathbf{R} \neq 0$ can not be induced by a Finsler metric.

However, Finsler metrics of scalar curvature are quite special in higher dimensions. This leads to the notion of flag curvature. Let $P \subset T_xM$ be a tangent plane and $y \in P \setminus \{0\}$. The pair $\{P, y\}$ is called a *flag* in T_xM. Then $P = \mathrm{span}\{y, u\}$, where $u \in P$ is an arbitrary vector linearly independent of y. Let L be a Finsler metric on an n-dimensional manifold M. It follows from (8.57) that the following quotient is independent of the choice of a particular u.

$$\mathbf{K}(P, y) := \frac{g_y(\mathbf{R}_y(u), u)}{g_y(y, y)g_y(u, u) - g_y(y, u)g_y(y, u)}. \tag{8.64}$$

The quantity $\mathbf{K}(P, y)$ is called the *flag curvature* of the flag $\{P, y\}$. There is a special class of Finsler metrics L such that for any $y \in TM \setminus \{0\}$, the flag curvature $\mathbf{K}(P, y)$ is independent of the tangent planes P containing y. As matter of fact, this class is quite rich. For a Finsler surface (M, L), the tangent plane P at each point is the whole tangent space T_xM. Thus by definition, F is contained in this class. In this case, $\mathbf{K}(y) := \mathbf{K}(T_xM, y)$ is a scalar function on $TM \setminus \{0\}$. We call it the *Gauss curvature.*

Clearly, a Finsler metric is of scalar curvature $\mathbf{K}(y)$ if and only if for any $y \in TM \setminus \{0\}$ the flag curvature $\mathbf{K}(P, y)$ is independent of the tangent planes P containing y. In particular, $\mathbf{R}_y = 0$ if and only if $\mathbf{K}(P, y) = 0$. That is, a Finsler metric is of zero curvature if and only if it is R-flat.

We have an analogue of Proposition 8.1.6 for Finsler metrics, which is due to L. Berwald

Proposition 8.2.4 ([Bw2]) *A Finsler metric is locally Minkowskian if and only if* $\mathbf{B} = 0$ *and* $\mathbf{R} = 0$.

Proof: Let (M, L) be a Finsler space. Assume that L is locally Minkowskian. Then there is a local coordinate system (x^i, y^i) in which $G^i = 0$, hence $\mathbf{B} = 0$ and $\mathbf{R} = 0$. Conversely, if $\mathbf{B} = 0$ and $\mathbf{R} = 0$, then $\Gamma^i_{jk} = \Gamma^i_{jk}(x)$ are functions of x only. The Berwald connection ∇ is affine with vanishing curvature. Therefore, at every point, there is a local coordinate system in which $\Gamma^i_{jk} = 0$. Hence $G^i = 0$. By (4.29), we have

$$L_{x^k y^l} y^k - L_{x^l} = 0. \tag{8.65}$$

Differentiating (8.65) with respect to y^m gives

$$L_{x^k y^l y^m} y^k + L_{x^m y^l} - L_{x^l y^m} = 0. \tag{8.66}$$

Contracting (8.66) with y^l yields

$$2L_{x^m} = 0.$$

Thus

$$L(y^1, \cdots, y^n) := L\Big(y^i \frac{\partial}{\partial x^i}\Big|_x\Big)$$

is independent of x, i.e., L is locally Minkowskian. We have proved the proposition. Q.E.D.

Remark 8.2.5 E. Cartan mentioned Proposition 8.2.4 in his monograph [Ca]. One can find a proof in [Nu]. See also [BaChSh1].

Example 8.2.2 Consider the following Finsler metric

$$L = \Big[3a(x,z)uw^2 + b(y)v^3\Big]^{2/3}.$$

According to Example 6.1.3, L is a Berwald metric. The coefficients of the Riemann curvature $\mathbf{R}$ are given by

$$R^1_1 = -\varphi\, uw, \qquad R^1_2 = 0, \qquad R^1_3 = \varphi\, u^2,$$

$$R^2_1 = 0, \qquad R^2_2 = 0, \qquad R^2_3 = 0$$

$$R^3_1 = \frac{1}{2}\varphi\, w^2, \qquad R^3_2 = 0, \qquad R^3_3 = -\frac{1}{2}\varphi\, uw,$$

where

$$\varphi := a_{xz}(x,z) - \frac{a_x(x,z)}{a(x,z)}\frac{a_z(x,z)}{a(x,z)}.$$

Thus $\mathbf{R} = 0$ if and only if $\varphi = 0$. ♯

Definition 8.2.6 Let L be a Finsler metric on n-manifold M. L is called an *Einstein* metric if there is a constant λ such that the Ricci scalar $R := \frac{1}{n-1}\mathbf{Ric}$ satisfies

$$R(y) = \lambda L(y), \qquad y \in TM. \tag{8.67}$$

L is said to be *Ricci-constant* if

$$R_{;i} := R_{x^i} - N^j_i R_{y^j} = 0. \tag{8.68}$$

L is said to be *weakly Ricci-constant* if

$$y^i R_{;i} := y^i R_{x^i} - 2G^i R_{y^i} = 0. \tag{8.69}$$

It follows from Lemma 4.2.2 that every Einstein metric is Ricci-constant.

Example 8.2.3 Consider the following two-dimensional spray

$$\mathbf{G} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - 3v\sqrt{u^2+v^2}\frac{\partial}{\partial u} - \frac{-u^4+u^2v^2+2v^4}{u\sqrt{u^2+v^2}}\frac{\partial}{\partial v}. \tag{8.70}$$

By (8.14), a direct computation yields

$$\begin{aligned} R_1^1 &= \frac{3}{2}\Big(3u^2+2v^2\Big)\Big(\frac{v}{u}\Big)^2 \\ R_2^2 &= -\frac{3}{2}\Big(3u^2+2v^2\Big)\Big(\frac{v}{u}\Big)^2. \end{aligned}$$

Thus the Ricci curvature

$$R = R_1^1 + R_2^2 = 0.$$

By the above argument, we conclude that this spray can not be induced by any singular Finsler metric.

♯

Example 8.2.4 Consider the following affine spray on R^2

$$\mathbf{G} := u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - \phi(x,y)u^2\frac{\partial}{\partial u} - \psi(x,y)v^2\frac{\partial}{\partial v}. \tag{8.71}$$

Using (8.14), one obtains

$$\begin{aligned} R_1^1 &= -uv\phi_y(x,y), \\ R_2^2 &= -uv\psi_x(x,y), \\ R_1^2 &= v^2\psi_x(x,y), \\ R_2^1 &= u^2\phi_y(x,y). \end{aligned}$$

The Ricci curvature is given by

$$\mathbf{Ric} = -uv\Big(\phi_y(x,y) + \psi_x(x,y)\Big). \tag{8.72}$$

Assume that ϕ and ψ are related by

$$\phi_y(x,y) = -\psi_x(x,y) \neq 0. \tag{8.73}$$

For example,

$$\phi := xy + \frac{1}{2}y^2, \qquad \psi := -\frac{1}{2}x^2 - xy.$$

Let $f(z)$ be a non-trivial analytic function on $\mathbb{C}$. Express f by

$$f(x+iy) = \varphi(x,y) + i\psi(x,y).$$

Note that ϕ and ψ also satisfy (8.73). For the above choice of ϕ and ψ, $\mathbf{G}$ is not induced by any Finsler metric. ♯

Example 8.2.5 ([Bw5]) Let $L = L(x, y, u, v)$ be a Finsler metric on an open subset $\mathcal{U} \subset \mathrm{R}^2$ in the form

$$L = e^{2\rho(x,y)} \exp\Big[2Q \arctan\Big(\frac{u}{v}\Big)\Big]\Big(u^2 + v^2\Big), \tag{8.74}$$

where $Q \geq 0$ is a constant. This metric was studied by P. Antonelli [An1] in ecology. The main scalar of L satisfies

$$\mathbf{I}^2 = \frac{4Q^2}{1+Q^2} < 4. \tag{8.75}$$

The spray coefficients $G = G^1$ and $H = G^2$ of the spray induced by L are given by

$$\begin{aligned} G &= \frac{1}{2(1+Q^2)}\Big\{(\rho_x - Q\rho_y)u^2 + 2(\rho_y + Q\rho_x)uv - (\rho_x - Q\rho_y)v^2\Big\} \\ H &= \frac{1}{2(1+Q^2)}\Big\{ - (\rho_y + Q\rho_x)u^2 + 2(\rho_x - 2Q\rho_y)uv + (\rho_y + Q\rho_x)v^2\Big\}. \end{aligned}$$

Thus L is indeed a Berwald metric.

Next we are going to compute the Gauss curvature of L. By (8.42) or (8.44), we immediately obtain a very simple formula for the Ricci scalar $R = \mathbf{Ric}$

$$R = -\frac{\rho_{xx} + \rho_{yy}}{1+Q^2}\Big(u^2 + v^2\Big). \tag{8.76}$$

Thus $R = 0$ if and only if

$$\rho_{xx} + \rho_{yy} = 0.$$

In this case, L is locally Minkowskian by Proposition 8.2.4. For example, when $\rho = ax + by + c$, $R = 0$. Hence L is locally Minkowskian. ♯

Example 8.2.6 ([Bw5]) Let $L = L(x, y, u, v)$ be a Finsler metric on an open subset $\mathcal{U} \subset \mathrm{R}^2$ in the form

$$L = e^{2\rho(x,y)}\Big(\frac{v^{1+a}}{u^a}\Big)^2, \tag{8.77}$$

where $a > 0$ is a positive constant. The main scalar of L satisfies

$$\mathbf{I}^2 = \frac{(1+2a)^2}{a(1+a)} > 4. \tag{8.78}$$

The spray coefficients $G = G^1$ and $H = G^2$ of the spray induced by L are given by

$$\begin{aligned} G &= -\frac{1}{2a}\rho_x u^2 \\ H &= \frac{1}{2a}\rho_y v^2. \end{aligned}$$

Thus L is a Berwald metric. By (8.42) or (8.44), we immediately obtain a very simple formula for the Ricci scalar

$$R = \frac{\rho_{xy}}{a(1+a)} uv. \tag{8.79}$$

Thus $R = 0$ if and only if $\rho_{xy} = 0$. In this case, L is locally Minkowskian. ♯

Example 8.2.7 Let $L = L(x, y, u, v)$ be a Finsler metric on an open subset $\mathcal{U} \subset \mathrm{R}^2$ in the form

$$L = e^{2\rho(x,y)} v^2 \exp\Big(2a\frac{u}{v}\Big), \tag{8.80}$$

where a is a constant. The main scalar of L satisfies

$$\mathbf{I}^2 = 4. \tag{8.81}$$

The spray coefficients $G = G^1$ and $H = G^2$ of the spray induced by L are given by

$$\begin{aligned} G &= -a\rho_x uv + \frac{1}{2}\Big(\rho_x - a\rho_y\Big)v^2 \\ H &= -\frac{1}{2}a\rho_x v^2 \end{aligned}$$

Thus L is a Berwald metric. By (8.42) or (8.44), we immediately obtain a very simple formula for the Ricci scalar

$$R = \rho_{xx} v^2. \tag{8.82}$$

Thus $R = 0$ if and only if $\rho_{xx} = 0$. In this case, L is locally Minkowskian. ♯

There are many two-dimensional Finsler metrics with $\mathbf{K} = 0$ which are not locally Minkowskian.

Proposition 8.2.7 *Let $L = L(x, y, u, v)$ be a Finsler metric on $\mathcal{U} \subset \mathrm{R}^2$ in the following form*

$$L = \Big[u\phi\Big(x, y, \frac{v}{u}\Big)\Big]^2,$$

where $\phi = \phi(x, y, \xi)$ is a positive function with $\phi_{\xi\xi} \neq 0$. Suppose that ϕ satisfies

$$\phi_y = 0, \qquad \frac{\phi_x}{\phi^2} + \Big(\frac{\phi_{x\xi}}{\phi\phi_{\xi\xi}}\Big)_\xi = 0. \tag{8.83}$$

Then $\mathbf{K} = 0$, hence $\mathbf{R} = 0$.

Proof: It suffices to prove that the Ricci curvature of F vanishes. Let

$$\mathbf{G} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - 2G\frac{\partial}{\partial u} - 2H\frac{\partial}{\partial v}$$

be the Finsler spray of F. We can express G and H by

$$\begin{aligned} G &= \frac{1}{2}\Theta\Big(x, y, \frac{v}{u}\Big)u^2 \\ H &= \frac{1}{2}\Theta\Big(x, y, \frac{v}{u}\Big)uv - \frac{1}{2}\Phi\Big(x, y, \frac{v}{u}\Big)u^2 \end{aligned}$$

where $\Theta = \Theta(x, y, \xi)$ and $\Phi = \Phi(x, y, \xi)$ are functions on $\mathcal{U} \times \mathrm{R}$. Assume that $\phi_y = 0$. Then By (6.51) and (6.52), we obtain

$$\Phi = -\frac{\phi_{x\xi}}{\phi_{\xi\xi}}, \qquad \Theta = \Phi\frac{\phi_\xi}{\phi} + \frac{\phi_x}{\phi}.$$

Thus

$$\Phi_y = 0.$$

One can easily verify that the second equation in (8.83) is equivalent to $\Theta = \Phi_\xi$. It follows from (8.47) that the Ricci curvature in the direction $(u, v) = (1, \xi)$

$$R = -\Phi_y = 0.$$

Therefore, $\mathbf{K} = 0$ by (8.63). Q.E.D.

Remark 8.2.8 There is another proof for Proposition 8.2.7. For any solution ϕ of (8.83), $\Phi = -\phi_{x\xi}/\phi_{\xi\xi}$ satisfies (8.50),

$$\Phi_y = 0, \qquad \Phi_{x\xi\xi} + \Phi\Phi_{\xi\xi\xi} = 0.$$

Thus $\mathbf{R} = 0$ by Proposition 8.1.11. Note that for Finsler metrics satisfying (8.83), the following four conditions are equivalent

(a) $\Phi_{\xi\xi\xi} = 0$;

(b) $\mathbf{J} = 0$;

(c) $\mathbf{E} = 0$;

(d) $\mathbf{B} = 0$.

If any of the above conditions is satisfied, then the resulting Finsler metric L is locally Minkowskian by Proposition 8.2.4. A natural question arises: Is there any R-flat Landsberg metric which is not locally Minkowskian? The question still remains open. According to Akbar-Zedah [AZ2], any positive definite R-flat Landsberg metric on a compact manifold must be locally Minkowskian. See Theorem 10.3.7 for a more general statement.

A trivial solution of (8.83) is

$$\phi(x,\xi) = h(x)(a+b\xi)^{\alpha}. \tag{8.84}$$

For this solution, the corresponding Finsler metric has constant main scalar. Moreover, the Berwald curvature vanishes. Thus F is actually locally Minkowskian.

If ϕ is a solution of (8.83) for which $\Phi = -\phi_{x\xi}/\phi_{\xi\xi}$ is not a polynomial of degree three or less, then the resulting Finsler metric L is not a Douglas metric (see Chapter 13 for definition). Hence L is not locally projectively flat, namely, there is no local coordinate system in which geodesics are straight lines.

8.3 Riemann Curvature of Riemannian Metrics

Riemannian spaces are very important Finsler spaces. Euclidean spaces are the simplest Riemannian spaces. Submanifolds in an Euclidean space are also Riemannian spaces. A. Einstein used Riemannian geometry to introduce his general relativity theory. Therefore, Riemannian geometry becomes one of the main streams in mathematics.

Let (M,g) be a Riemannian space. The Riemannian metric g can be viewed as a special Finsler metric by setting

$$L(y) := g(y,y).$$

L is a positive definite Finsler metric, so that we usually denote it by

$$F(y) := \sqrt{g(y,y)}.$$

The spray $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}$ induced by g is given by

$$G^i(y) = \frac{1}{4} g^{il}\Big\{2\frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\Big\} y^j y^k. \tag{8.85}$$

Clearly, $\mathbf{G}$ is affine, namely, $G^i(y)$ are quadratic in $y \in T_xM$. The Christoffel symbols of $\mathbf{G}$

$$\Gamma^i_{jk}(x) := \frac{\partial^2 G^i}{\partial y^j \partial y^k}(y) = \frac{1}{2} g^{il}\Big\{\frac{\partial g_{kl}}{\partial y^j} + \frac{\partial g_{jl}}{\partial y^k} - \frac{\partial g_{jk}}{\partial x^l}\Big\}$$

are functions of x only. In this case, the Berwald connection ∇ is affine. It coincides with the N-connection D. We call it the *Levi-Civita connection* in Riemannian geometry. D is expressed by

$$\mathrm{D}_u V = \Big\{dV^i(u) + v^j \Gamma^i_{jk}(x) u^k\Big\}\frac{\partial}{\partial x^i}\Big|_x,$$

where $u, v \in T_xM$ and $V \in C^\infty(TM)$ with $V_x = v$. For a vector $y \in T_xM \setminus \{0\}$, we express the Riemann curvature $\mathbf{R}_y : T_xM \to T_xM$ by

$$\mathbf{R}_y(u) = R^i_k(y) u^k \frac{\partial}{\partial x^i}|_x, \qquad u = u^i \frac{\partial}{\partial x^i}|_x \in T_xM.$$

Note $R^i_k(y)$ are quadratic in $y \in T_xM$. Set

$$R_j{}^i{}_{kl} = \frac{1}{3}\Big\{ \frac{\partial^2 R^i_k}{\partial y^j \partial y^l} - \frac{\partial^2 R^i_l}{\partial y^j \partial y^k} \Big\}.$$

$R_j{}^i{}_{kl}$ live on M. Thus we obtain a tensor R on M

$$\mathrm{R}(u,v)w := R_j{}^i{}_{kl}(x) u^k v^l w^j \frac{\partial}{\partial x^i}|_x.$$

R is called the *Riemann curvature tensor.*

A direct computation yields

$$R_j{}^i{}_{kl}(x) = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^i_{ks}\Gamma^s_{jl} - \Gamma^s_{jk}\Gamma^i_{ls}, \tag{8.86}$$

and

$$R^i_k(y) = R_j{}^i{}_{kl}(x) y^j y^l. \tag{8.87}$$

In the index-free form

$$\mathrm{R}(U,V)W = \mathrm{D}_U \mathrm{D}_V W - \mathrm{D}_V \mathrm{D}_U W - \mathrm{D}_{[U,V]}W, \tag{8.88}$$

where $U, V, W \in C^\infty(TM)$ and

$$\mathbf{R}_y(u) = \mathrm{R}(u,y)y.$$

R has the following properties

$$\begin{aligned}
&\mathrm{R}(u,v)w + \mathrm{R}(v,u)w = 0,\\
&\mathrm{R}(u,v)w + \mathrm{R}(v,w)u + \mathrm{R}(w,u)v = 0,\\
&g(\mathrm{R}(u,v)w,z) - g(\mathrm{R}(w,z)u,v) = 0,\\
&g(\mathrm{R}(u,v)w,z) + g(\mathrm{R}(u,v)z,w) = 0.
\end{aligned}$$

The above identities imply

$$g(\mathbf{R}_y(u),v) = g(u,\mathbf{R}_y(v)) \tag{8.89}$$

and

$$g(\mathbf{R}_y(u),u) = g(\mathbf{R}_u(y),y). \tag{8.90}$$

By (8.90), we can see that for a 2-dimensional plane $P = \mathrm{span}\{y,u\} \subset T_xM$, the flag curvature $\mathbf{K}(P,y)$

$$\mathbf{K}(P,y) := \frac{g(\mathbf{R}_y(u),u)}{g(y,y)g(u,u) - g(y,u)^2}$$

is independent of $y \in P$. We call $\mathbf{K}(P) := \mathbf{K}(P,y)$ the *sectional curvature* of the "section" $P \subset T_xM$. A Riemannian metric g is of constant curvature λ if the sectional curvature $\mathbf{K}(P) = \lambda$ for any section $P \subset T_xM$. Locally, for number $\lambda \in \mathrm{R}$, there is only one Riemannian metric of constant curvature λ up to an isometry. This fact is due to E. Cartan.

Consider a Minkowski metric L on a vector space V. L induces a Riemannian metric $\hat{g}$ on $\mathrm{V} \setminus \{0\}$. Let $\mathbf{C}_y : T_y\mathrm{V} \times T_yV \to T_y\mathrm{V}$. The Riemann curvature $\hat{\mathbf{R}}$ of $\hat{g}$ are given by

$$\hat{\mathbf{R}}(u,v)w = \mathbf{C}_y(v, \mathbf{C}_y(u,w)) - \mathbf{C}_y(u, \mathbf{C}_y(v,w)), \tag{8.91}$$

where $u, v, w \in T_y\mathrm{V} \approx \mathrm{V}$. See [KawA]. According to a theorem of Brickell [Bk1][Sc], a reversible Minkowski functional on V of $\dim \mathrm{V} \geq 3$ must be Euclidean if $\hat{\mathbf{R}} = 0$.

The indicatrix $\mathrm{S} = F^{-1}(1)$ is a hypersurface in V. Let $\dot{g}$ denote the induced Riemannian metric on S. We have the following

Proposition 8.3.1 *The Riemann curvature $\dot{\mathbf{R}}$ of $\dot{g}$ has the following form*

$$\dot{\mathbf{R}}(u,v)w = \mathbf{C}_y(\mathbf{C}_y(u,w), v) - \mathbf{C}_y(\mathbf{C}_y(v,w), u) + \dot{g}_y(v,w)u - \dot{g}_y(u,w)v, \tag{8.92}$$

where $u, v, w \in T_y\mathrm{S} \approx \mathrm{W}_y \subset \mathrm{V}$.

The proof is left to the reader.

Example 8.3.1 (H. Kawaguchi [KawH]) Let L_o be an 2-dimensional Minkowski functional on R^2 and $|\cdot|$ be the canonical Euclidean norm on R^n. Define

$$L(u,v) := L_o(u) + |v|^2, \qquad (u,v) \in \mathrm{R}^2 \times \mathrm{R}^n = \mathrm{R}^{n+2}.$$

Then L is a singular Minkowski functional. Further, the induced Riemannian metric $\hat{g}$ on R^{n+2} is always flat. ♯

Example 8.3.2 (G.S. Asanov [As2][As3]). Let $|\cdot|$ denote the canonical Euclidean norm on R^n. Define a homogeneous functional B on $\mathrm{R}^{n+1} = \mathrm{R} \times \mathrm{R}^n$ by

$$B(y^0, y) := |y|^2 + C_1|y^0| \cdot |y| + C_2|y^0|^2, \qquad (y^0, y) \in \mathrm{R} \times \mathrm{R}^n. \tag{8.93}$$

Let

$$D := (C_1)^2 - 4C_2.$$

Assume that $D < 0$. Then B is positive-definite. Define a functional $L : \mathrm{R}^{n+1} \to [0, \infty)$ by

$$L(y^0, y) := B(y^0, y) \exp\Big\{2C_1\sqrt{-D}A(y^0, y)\Big\}, \tag{8.94}$$

where

$$A(y^0, y) := \tan^{-1}\Big(\frac{\sqrt{-D}|y^0|}{C_1|y^0| + 2|y|}\Big) - \tan^{-1}\Big(\frac{\sqrt{-D}}{C_1}\Big)$$

G.S. Anasov proved that $(g_{ij}(y^0, y))$ is positive definite. Further, the induced Riemannian metric $\dot{g}$ on the indicatrix has constant curvature $\frac{\sqrt{-D}}{4C_1}$. ♯

8.4 Geodesic Fields and Riemann Curvature

A natural question arises: given two sprays on a manifold, how closely related they are so that the Riemann curvature are equal in certain directions ? In this section, we will discuss this problem.

Let $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}$ be a spray on a manifold M. Recall that a non-zero vector field $Y = Y^i \frac{\partial}{\partial x^i}$ is geodesic if

$$Y^j \frac{\partial Y^i}{\partial x^j} + 2G^i(Y) = 0. \tag{8.95}$$

In other words, all the integral curves of Y are geodesics of $\mathbf{G}$.

Definition 8.4.1 Let $\mathbf{G}$ and $\hat{\mathbf{G}}$ be sprays on M and Y a non-zero vector field on M. They are said to be Y*-related* if the corresponding N-connections D and $\hat{\mathrm{D}}$ are equal in the direction Y, $\mathrm{D}_Y = \hat{\mathrm{D}}_Y$, i.e., their connection coefficients are related by

$$N^i_j(Y) = \hat{N}^i_j(Y). \tag{8.96}$$

Let $\mathbf{G}$ and $\hat{\mathbf{G}}$ be two Y-related sprays. Suppose that Y is geodesic with respect to $\mathbf{G}$. It follows from (8.96) that

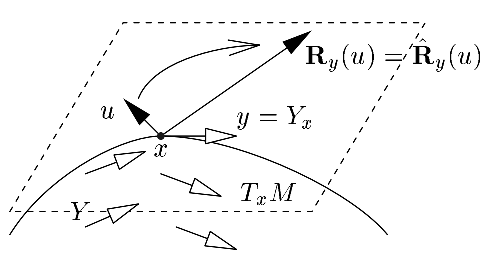
$$G^i(Y) = N^i_j(Y)Y^j = \hat{N}^i_j(Y)Y^j = \hat{G}^i(Y). \tag{8.97}$$

Thus Y is also geodesic with respect to $\hat{\mathbf{G}}$.

The following theorem is very useful in comparison geometry.

Proposition 8.4.2 *Let Y be a geodesic field on a spray space $(M, \mathbf{G})$. Assume that a spray $\hat{\mathbf{G}}$ on M is Y-related to $\mathbf{G}$. Then for $y = Y_x \in T_xM$*

$$\mathbf{R}_y = \hat{\mathbf{R}}_y. \tag{8.98}$$



Proof: Observe that

$$\begin{aligned}\frac{\partial \hat{G}^i}{\partial x^k}(Y) &= \frac{\partial}{\partial x^k}\Big[\hat{G}^i(Y)\Big] - \hat{N}^i_j(Y)\frac{\partial Y^j}{\partial x^k}\\ &= \frac{\partial}{\partial x^k}\Big[G^i(Y)\Big] - N^i_j(Y)\frac{\partial Y^j}{\partial x^k}\\ &= \frac{\partial G^i}{\partial x^k}(Y) \qquad (8.99)\end{aligned}$$

and

$$\begin{aligned}Y^j\frac{\partial \hat{N}^i_k}{\partial x^j}(Y) &= Y^j\Big\{\frac{\partial}{\partial x^j}\Big[\hat{N}^i_k(Y)\Big] - \hat{\Gamma}^i_{kl}(Y)\frac{\partial Y^l}{\partial x^j}\Big\}\\ &= Y^j\frac{\partial}{\partial x^j}\Big[N^i_k(Y)\Big] + 2\hat{G}^l(Y)\hat{\Gamma}^i_{kl}(Y)\\ &= Y^j\frac{\partial N^i_k}{\partial x^j}(Y) + \Gamma^i_{kl}(Y)Y^j\frac{\partial Y^l}{\partial x^j} + 2\hat{G}^l(Y)\hat{\Gamma}^i_{kl}(Y)\\ &= Y^j\frac{\partial N^i_k}{\partial x^j}(Y) - 2G^l(Y)\Gamma^i_{kl}(Y) + 2\hat{G}^l(Y)\hat{\Gamma}^i_{kl}(Y).\end{aligned}$$

This implies

$$Y^j\frac{\partial \hat{N}^i_k}{\partial x^j}(Y) - 2\hat{G}^l(Y)\hat{\Gamma}^i_{kl}(Y) = Y^j\frac{\partial N^i_k}{\partial x^j}(Y) - 2G^l(Y)\Gamma^i_{kl}(Y). \qquad (8.100)$$

Plugging (8.99) and (8.100) into (8.11) yields

$$\hat{R}^i_k(Y) = R^i_k(Y).$$

Q.E.D.

Now we prove the main result of this section.

Proposition 8.4.3 *Let (M, L) be a Finsler space. For any $y \in T_xM$,*

$$\mathbf{R}_y = \hat{\mathbf{R}}_y, \qquad (8.101)$$

where $\hat{\mathbf{R}}$ is the Riemann curvature of $\hat{g} := g_Y$ and Y is a geodesic field of L with $Y_x = y$.

Proof: Let $\hat{\mathrm{D}}$ denote the Levi-Civita connection of $\hat{g}$. By Lemma 7.1.4,

$$\mathrm{D}_Y = \hat{\mathrm{D}}_Y. \qquad (8.102)$$

Thus L is Y-related to $\hat{g}$ and (8.101) follows from Proposition 8.4.2. Q.E.D.

Remark 8.4.4 Assuming that Proposition 8.2.1 holds for Riemannian metrics, then it also holds for Finsler metrics by a simple argument using Proposition 8.4.3. Let (M, L) be a Finsler space. For a vector $y \in T_xM \setminus \{0\}$, extend y to a geodesic field Y in a neighborhood $\mathcal{U}_x$. Let $\hat{\mathbf{R}}$ denote the Riemann curvature of $\hat{g} := g_Y$. By assumption, for any $z \in \mathcal{U}$ and $w \in T_zM \setminus \{0\}$, the following holds

$$\hat{g}\Big(\hat{\mathbf{R}}_w(u), v\Big) = \hat{g}\Big(u, \hat{\mathbf{R}}_w(v)\Big), \qquad u, v \in T_zM.$$

Restricting the above equation to x with $w = y$, we obtain (8.57).

8.5 Variational Formulas

For Finsler metrics, the geometric meaning of the Riemann curvature lies in the second variation formulas. In this section, we will give the second variational formulas for Finsler metrics.

Let (M, L) be a Finsler space. Consider the following variational problem

$$\mathcal{E}(c) := \int_a^b L(\dot{c}(t))dt = \text{Extremum},$$

where $c : [a, b] \to M$ is a C^∞ curve. Take a variation of c,

$$H : (-\varepsilon, \varepsilon) \times [a, b] \to M$$

with fixed endpoints, i.e.,

$$H(0, t) = c(t), \quad H(u, a) = c(a), \quad H(u, b) = c(b).$$

Put

$$\mathcal{E}(u) := \int_a^b L\Big(\frac{\partial H}{\partial t}(u, t)\Big)dt.$$

Assume c is a geodesic. Hence $\mathcal{E}'(0) = 0$. Let $V(t) := \frac{\partial H}{\partial u}(0, t)$. A direct computation yields the second variation formula

$$\mathcal{E}''(0) = 2\int_a^b \Big\{ g_{\dot{c}}(\mathrm{D}_{\dot{c}}V, \mathrm{D}_{\dot{c}}V) - g_{\dot{c}}(\mathbf{R}_{\dot{c}}(V), V)\Big\}dt. \tag{8.103}$$

Thus, the sign of the Riemann curvature tells us some properties of $\mathcal{E}$ nearby c. For example, if L is positive definite and the Riemann curvature satisfies

$$g_y(\mathbf{R}_y(v), v) \leq 0,$$

then for any variation H of c,

$$\mathcal{E}''(0) \geq 0.$$

Assume that L is positive definite. Let $F = \sqrt{L}$. We consider the following variational problem of length

$$\mathcal{L}(c) := \int_a^b F(\dot{c}(t))dt = \text{Extremum},$$

where $c : [a, b] \to M$ is a C^∞ curve. Assume that $c : [a, b] \to M$ be a constant speed curve of F with

$$F(\dot{c}(t)) = \lambda \neq 0.$$

Then the function

$$\mathcal{L}(u) := \int_a^b F\Big(\frac{\partial H}{\partial t}(u, t)\Big) dt$$

satisfies

$$\mathcal{L}''(0) = \frac{1}{\lambda} \int_a^b \Big\{ g_{\dot{c}}(\mathrm{D}_{\dot{c}} V^\perp, \mathrm{D}_{\dot{c}} V^\perp) - g_{\dot{c}}(\mathbf{R}_{\dot{c}}(V^\perp), V^\perp) \Big\} dt, \tag{8.104}$$

where $V^\perp(t)$ is given by

$$V^\perp(t) := V(t) - \frac{1}{\lambda^2} g_{\dot{c}(t)}(\dot{c}(t), V(t))\ \dot{c}(t).$$

By (8.104), we can prove the Synge Theorem: for an even-dimensional oriented closed positive definite Finsler space (M, F), if the flag curvature is positive, then the fundamental group $\pi_1(M) = 0$. See [BaChSh1].

Chapter 9

Structure Equations of Sprays

Sprays are special vector fields on the tangent bundle. All of the quantities defined by a spray live on the tangent bundle. In previous chapters, we treat them as quantities on the base manifold by choosing a reference vector. To find the internal relationship among various quantities such as the Berwald curvature, Landsberg curvature and Riemann curvature, etc., one has to go up to the tangent bundle. It seems that exterior differential method is quite useful in computation for this purpose. We will employ it to do some complicated computations.

9.1 Berwald Connection Forms

Let M be an n-dimensional manifold. Let (x^i, y^i) be a standard local coordinate system in TM. The *vertical tangent bundle* of $TM \setminus \{0\}$ is defined by

$$\mathcal{V}TM := \mathrm{span}\Big\{ \frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^n} \Big\}$$

and the *canonical horizontal co-tangent bundle* of TM is defined by

$$\mathcal{H}^*TM := \mathrm{span}\Big\{ dx^1, \cdots, dx^n \Big\}.$$

These are two special vector bundles over TM which are independent of any geometric structure.

Now we consider a spray $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}$ on M. We are going to show that it determines the complements of $\mathcal{V}TM$ and $\mathcal{H}^*TM$. Let

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}, \qquad \delta y^i := dy^i + N^i_j dx^j,$$

where $N^i_j := \frac{\partial G^i}{\partial y^j}$. Put

$$\mathcal{H}TM := \mathrm{span}\Big\{\frac{\delta}{\delta x^1}, \cdots, \frac{\delta}{\delta x^n}\Big\}, \qquad \mathcal{V}^*TM := \mathrm{span}\Big\{\delta y^1, \cdots, \delta y^n\Big\}.$$

One can easily show that $\mathcal{H}TM$ and $\mathcal{V}^*TM$ are well-defined vector bundles over $TM \setminus \{0\}$. We obtain a direct decomposition for $T(TM \setminus \{0\})$

$$T(TM \setminus \{0\}) = \mathcal{H}TM \oplus \mathcal{V}TM,$$

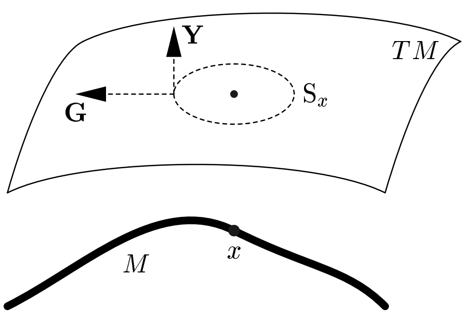
and a direct decomposition for $T^*(TM \setminus \{0\})$

$$T^*(TM \setminus \{0\}) = \mathcal{H}^*TM \oplus \mathcal{V}^*TM.$$

There are two canonical sections of $\mathcal{V}TM$ and $\mathcal{H}TM$.

$$\mathbf{Y} := y^i \frac{\partial}{\partial y^i}, \qquad \mathbf{G} := y^i \frac{\delta}{\delta x^i}. \tag{9.1}$$

In this sense, $\mathbf{G}$ is a horizontal vector field on $TM \setminus \{0\}$.



The above setting is in a standard local coordinate system in TM. Now we turn to an arbitrary local (co-)frame to gain more freedom. Take an arbitrary local coframe $\{\omega^i\}_{i=1}^n$ for $\mathcal{H}^*TM$ and the corresponding coframe $\{\omega^{n+i}\}_{i=1}^n$ for $\mathcal{V}^*TM$. Express them by

$$\omega^i := a^i_j dx^j, \qquad \omega^{n+i} := a^i_j \delta y^j \tag{9.2}$$

where (a^i_j) is a local $n \times n$ matrix-valued function on $TM \setminus \{0\}$. Let $\Big(b^i_j\Big) := \Big(a^i_j\Big)^{-1}$. The dual frame $\{e_i\}_{i=1}^n$ for $\mathcal{V}TM$ and $\{e_{n+i}\}_{i=1}^n$ for $\mathcal{H}TM$ are given by

$$e_i = b^j_i \frac{\partial}{\partial y^j}, \qquad e_{n+i} = b^j_i \frac{\delta}{\delta x^j}.$$

The Berwald connection can be defined as a linear connection on $\mathcal{V}TM$. Let

$$\omega_j^{\ i} := a_k^i \Big\{ db_j^k + b_j^m \Gamma^k_{ml} dx^l \Big\},$$

where $\Gamma^i_{jk} = \frac{\partial^2 G^i}{\partial y^j \partial y^k}$. The *Berwald connection* ∇ on $\mathcal{V}TM$ is defined by

$$\nabla X := dX^i \otimes e_i + X^j \omega_j^{\ i} \otimes e_i, \tag{9.3}$$

where $X = X^i e_i \in C^\infty(\mathcal{V}TM)$. We call $\{\omega_j^{\ i}\}_{ij=1}^n$ the *Berwald connection form* with respect to $\{\omega^i\}_{i=1}^n$. The co-frame $\{\omega^{n+i}\}_{i=1}^n$ can be expressed as

$$\omega^{n+i} = dv^i + v^j \omega_j^{\ i}, \tag{9.4}$$

where $v^i = a^i_j y^j$ are the coefficients of $\mathbf{Y} = v^i e_i$.

9.2 Curvature Forms and Bianchi Identities

As we have seen, the Cartan connection and the Chern connection are defined only for Finsler spaces, while the Berwald connection is defined for general spray spaces. Therefore, we will use the Berwald connection as our main tool to study spray spaces and Finsler spaces.

Let $(M, \mathbf{G})$ be a spray space. Take an arbitrary local coframe $\{\omega^i\}_{i=1}^n$ for $\mathcal{H}^*TM$ and the corresponding coframe $\{\omega^{n+i}\}_{i=1}^n$ for $\mathcal{V}^*TM$. Let $\{\omega^i_j\}$ be the Berwald connection forms with respect to $\{\omega^i\}_{i=1}^n$. The symmetry $\Gamma^i_{jk} = \Gamma^i_{kj}$ is equivalent to

$$d\omega^i = \omega^j \wedge \omega_j^{\ i}. \tag{9.5}$$

Put

$$\Omega_j^{\ i} := d\omega_j^{\ i} - \omega_j^{\ k} \wedge \omega_k^{\ i}. \tag{9.6}$$

The matrix-valued 2-form $\Omega = (\Omega_j^{\ i})$ is called the *curvature form* of the Berwald connection with respect to $\{e_i\}$ or $\{\omega^i\}$. Differentiating (9.5) yields

$$0 = d^2\omega^i = d\omega^j \wedge \omega_j^{\ i} - \omega^j \wedge d\omega_j^{\ i} = \omega^j \wedge \Omega_j^{\ i}.$$

Thus $\Omega_j^{\ i}$ must be in the following form

$$\Omega_j^{\ i} = \frac{1}{2} R_j^{\ i}{}_{kl}\, \omega^k \wedge \omega^l - B^i_{jkl}\, \omega^k \wedge \omega^{n+l}, \tag{9.7}$$

where

$$R_j^{\ i}{}_{kl} + R_j^{\ i}{}_{lk} = 0, \tag{9.8}$$

$$R_j^{\ i}{}_{kl} + R_k^{\ i}{}_{lj} + R_l^{\ i}{}_{jk} = 0 \tag{9.9}$$

and

$$B^i_{jkl} = B^i_{kjl}. \tag{9.10}$$

Put

$$\Omega^i := d\omega^{n+i} - \omega^{n+j} \wedge \omega_j^{\ i}. \tag{9.11}$$

Differentiating (9.4) yields

$$\Omega^i = v^j \Omega_j^{\ i}. \tag{9.12}$$

Thus we can express Ω^i in the following form

$$\Omega^i = \frac{1}{2} R^i_{\ kl} \omega^k \wedge \omega^l - B^i_{kl} \omega^k \wedge \omega^{n+l}, \tag{9.13}$$

where

$$R^i_{\ kl} := v^j R_j^{\ i}{}_{kl}, \qquad B^i_{kl} := v^j B^i_{jkl}. \tag{9.14}$$

Put

$$R^i_k := v^l R^i_{\ kl} = v^j v^l R_j^{\ i}{}_{kl}. \tag{9.15}$$

As a matter of fact, the above quantities in (9.7) and (9.14) are not new. We will show that $R_j^{\ i}{}_{kl}$ are exactly the coefficients of the Riemann curvature tensor given in (8.20) and B^i_{jkl} are exactly the coefficients of the Berwald curvature given in (6.4). Moreover, $B^i_{kl} = 0$. First we prove the following

Lemma 9.2.1

$$d\Omega_j^{\ i} + \Omega_j^{\ k} \wedge \omega_k^{\ i} - \omega_j^{\ k} \wedge \Omega_k^{\ i} = 0. \tag{9.16}$$

Proof: The proof is direct, using (9.6).

$$\begin{aligned} d\Omega_j^{\ i} &= -d\omega_j^{\ k} \wedge \omega_k^{\ i} + \omega_j^{\ k} \wedge d\omega_k^{\ i} \\ &= -(\Omega_j^{\ k} + \omega_j^{\ l} \wedge \omega_l^{\ k}) \wedge \omega_k^{\ i} + \omega_j^{\ k} \wedge (\Omega_k^{\ j} + \omega_k^{\ l} \wedge \omega_l^{\ i}) \\ &= -\Omega_j^{\ k} \wedge \omega_k^{\ i} + \omega_j^{\ k} \wedge \Omega_k^{\ i}. \end{aligned}$$

Q.E.D.

Define $R_j^{\ i}{}_{kl;m}$ and $R_j^{\ i}{}_{kl\cdot m}$ by

$$\begin{aligned} dR_j^{\ i}{}_{kl} &- R_m^{\ i}{}_{kl} \omega_i^{\ m} - R_j^{\ i}{}_{ml} \omega_k^{\ m} - R_j^{\ i}{}_{km} \omega_l^{\ m} + R_j^{\ m}{}_{kl} \omega_m^{\ i} \\ &=: R_j^{\ i}{}_{kl;m} \omega^m + R_j^{\ i}{}_{kl\cdot m} \omega^{n+m}. \end{aligned} \tag{9.17}$$

Similarly, we define $B^i_{jkl;m}$ and $B^i_{jkl\cdot m}$ by

$$\begin{aligned} dB^i_{jkl} &- B^i_{mkl} \omega_i^{\ m} - B^i_{jml} \omega_k^{\ m} - B^i_{jkm} \omega_l^{\ m} + B^m_{jkl} \omega_m^{\ i} \\ &=: B^i_{jkl;m} \omega^m + B^i_{jkl\cdot m} \omega^{n+m}. \end{aligned} \tag{9.18}$$

From (9.16), one obtains the following Bianchi identities

$$R_j^{\ i}{}_{kl;m} + R_j^{\ i}{}_{lm;k} + R_j^{\ i}{}_{mk;l} = 0; \tag{9.19}$$

$$R_j^{\ i}{}_{kl\cdot m} = B^i_{jml;k} - B^i_{jkm;l}; \tag{9.20}$$

$$B^i_{jkl\cdot m} = B^i_{jkm\cdot l}. \tag{9.21}$$

Contracting (9.19) with v^j, we obtain a Bianchi identity for $R^i{}_{kl} = R_j{}^i{}_{kl} v^j$.

$$R^i{}_{kl;m} + R^i{}_{lm;k} + R^i{}_{mk;l} = 0. \tag{9.22}$$

Contracting (9.22) with v^l yields

$$R^i_{k;m} - R^i_{m;k} + R^i{}_{mk;l} v^l = 0. \tag{9.23}$$

Let $\{\omega^i := dx^i,\ \omega^{n+i} := \delta y^i\}$ be the natural local coframe for $\mathcal{H}^*TM \oplus \mathcal{V}^*TM$. We obtain

$$\Omega_j{}^i = \Big\{ \frac{\partial \Gamma^i_{jl}}{\partial x^k} + N^m_l \frac{\partial \Gamma^i_{jk}}{\partial y^m} - \Gamma^m_{jk}\Gamma^i_{ml} \Big\} dx^k \wedge dx^l - \frac{\partial \Gamma^i_{jk}}{\partial y^l} dx^k \wedge \delta y^l.$$

Thus

$$R_j{}^i{}_{kl} = \frac{\delta \Gamma^i_{jl}}{\delta x^k} - \frac{\delta \Gamma^i_{jk}}{\delta x^l} + \Gamma^m_{jl}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{ml}, \tag{9.24}$$

$$B^i_{jkl} = \frac{\partial \Gamma^i_{jk}}{\partial y^l} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}. \tag{9.25}$$

By the homogeneity of Γ^i_{jk} and (9.25), we conclude that

$$B^i_{jk} = B^i_{jkl} y^l = \frac{\partial \Gamma^i_{jk}}{\partial y^l} y^l = 0.$$

Thus (9.13) simplifies to

$$\Omega^i = \frac{1}{2} R^i{}_{kl} dx^k \wedge dx^l.$$

9.3 R-Quadratic and R-Flat Sprays

A spray is said to be R-quadratic if $\mathbf{R}_y$ is quadratic in y, namely, in local coordinates, $R^i_k(y)$ are quadratic in $y \in T_xM$, equivalently, $R_j{}^i{}_{kl}(y)$ are functions of x only. This is also equivalent to that $R_j{}^i{}_{kl\cdot m}(y) = 0$. Note that any affine spray ($\mathbf{B} = 0$) is R-quadratic. However, R-quadratic sprays are not necessarily Berwaldian as shown by Theorem 9.3.3. By definition, a spray is said to be R-flat if $\mathbf{R} = 0$. In this section, we will study the Berwald curvature of R-quadratic and R-flat sprays.

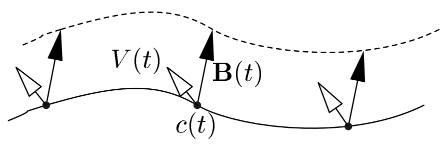
Proposition 9.3.1 *Let $(M, \mathbf{G})$ be a spray space. Suppose that $\mathbf{G}$ is R-quadratic. Then the Berwald curvature $\mathbf{B}$ is parallel along geodesics. More precisely, for any geodesic $c(t)$ in M and any parallel vector field $V(t)$ along c, the following vector field*

$$\mathbf{B}(t) := \mathbf{B}_{\dot c(t)}(V(t), V(t), V(t)) \tag{9.26}$$

is parallel along c, and the following function

$$\mathbf{E}(t) := \mathbf{E}_{\dot{c}(t)}(V(t), V(t)) \tag{9.27}$$

is a constant along c.



Proof: In local coordinates, let $V(t) = V^i(t)\frac{\partial}{\partial x^i}$ and $\dot{c}(t) = \frac{dx^i}{dt}(t)\frac{\partial}{\partial x^i}$. We have

$$\begin{aligned} \mathrm{D}_{\dot{c}}\mathbf{B}(t) &= B^i_{jkl;p}\frac{dc^p}{dt}(t)V^j(t)V^k(t)V^l(t)\ \frac{\partial}{\partial x^i}|_{c(t)} && (9.28)\\ \mathbf{E}'(t) &= \frac{1}{2}B^m_{jml;p}\frac{dc^p}{dt}V^j(t)V^l(t). && (9.29)\end{aligned}$$

It suffices to prove that $B^i_{jkl;m}y^m = 0$. By assumption

$$R_j{}^i{}_{kl\cdot m} = \frac{\partial R_j{}^i{}_{kl}}{\partial y^m} = 0.$$

It follows from (9.20) that

$$B^i_{jml;k} = B^i_{jkm;l}. \tag{9.30}$$

Contracting (9.30) with y^p and using the fact $y^l B^i_{jkl} = 0$, we obtain

$$B^i_{jml;p}y^p = B^i_{jpm;l}y^p = (B^i_{jpm}y^p)_{;l} = 0. \tag{9.31}$$

Thus $\mathbf{B}(t)$ is parallel. From (9.31), we immediately obtain

$$E_{jl;p}y^p = \frac{1}{2}B^m_{jml;p}y^p = 0.$$

This implies that $\mathbf{E}(t) = constant$. Q.E.D.

For further study on the Berwald curvature, we need the Ricci identities for the Berwald curvature $\mathbf{B}$. Differentiating (9.18) yields

$$\begin{aligned}\frac{1}{2}\Big(B^i_{jkl;m;s} - B^i_{jkl;s;m}\Big)\omega^m\wedge\omega^s + \Big(B^i_{jkl;m\cdot s} - B^i_{jkl\cdot s;m}\Big)\omega^m\wedge\omega^{n+s}\\ +\frac{1}{2}\Big(B^i_{jkl\cdot m\cdot s} - B^i_{jkl\cdot s\cdot m}\Big)\omega^{n+m}\wedge\omega^{n+s}\\ = B^i_{jkl\cdot p}\Omega^p + B^i_{pkl}\Omega_j{}^p + B^i_{jpl}\Omega_k{}^p + B^i_{jkp}\Omega_l{}^p - B^p_{jkl}\Omega_p{}^i.\end{aligned}$$

We obtain

$$B^i_{jkl\cdot m;s} = B^i_{jkl;s\cdot m} - B^p_{jkl}B^i_{pms} + B^i_{pkl}B^p_{jms} + B^i_{jpl}B^p_{kms} + B^i_{jkp}B^p_{lms}. \tag{9.32}$$

$$B^i_{jkl;m;s} = B^i_{jkl;s;m} + B^i_{jkl\cdot p}R^p{}_{ms} - B^p_{jkl}R_p{}^i{}_{ms} + B^i_{pkl}R_j{}^p{}_{ms} + B^i_{jpl}R_k{}^p{}_{ms} + B^i_{jkp}R_l{}^p{}_{ms}. \tag{9.33}$$

For simplicity, let

$$\tilde{B}^i_{jklm} := B^i_{jkl\cdot m}, \qquad \bar{B}^i_{jklm} := B^i_{jkl;m}.$$

Define

$$\tilde{\mathbf{B}}_y(u,v,w,z) : = \tilde{B}^i_{jklm}(y)u^jv^kw^lz^m \frac{\partial}{\partial x^i}|_x,$$

$$\bar{\mathbf{B}}_y(u,v,w,z) : = \bar{B}^i_{jklm}(y)u^jv^kw^lz^m \frac{\partial}{\partial x^i}|_x.$$

These two quantities play an important role in the study of R-flat sprays.

Clearly, $\tilde{\mathbf{B}}_y(u,v,w,z)$ is symmetric in $u,v,w,z \in T_xM$. This fact follows from (9.21). From (9.20) we see that if $\mathbf{G}$ is R-quadratic, then $\bar{\mathbf{B}}_y(u,v,w,z)$ is symmetric in $u,v,w,z \in T_xM$, too.

Rewrite (9.32) and (9.33) as follows

$$\tilde{B}^i_{jklm;s} = \bar{B}^i_{jkls\cdot m} - B^p_{jkl}B^i_{pms} + B^i_{pkl}B^p_{jms} + B^i_{jpl}B^p_{kms} + B^i_{jkp}B^p_{lms}. \tag{9.34}$$

$$\bar{B}^i_{jklm;s} = \bar{B}^i_{jkls;m} + \tilde{B}^i_{jklp}R^p_{ms} - B^p_{jkl}R_p{}^i{}_{ms} + B^i_{pkl}R_j{}^p{}_{ms} + B^i_{jpl}R_k{}^p{}_{ms} + B^i_{jkp}R_l{}^p{}_{ms}. \tag{9.35}$$

First we assume that $\mathbf{G}$ is R-quadratic. Rewrite (9.31) as

$$\bar{B}^i_{jklm}y^m = 0. \tag{9.36}$$

(9.36) implies

$$\bar{B}^i_{jkls\cdot m}y^s = [\bar{B}^i_{jkls}y^s]_{\cdot m} - \bar{B}^i_{jklm} = -\bar{B}^i_{jklm}. \tag{9.37}$$

Contracting (9.34) with y^s and using (9.37), we obtain

$$\tilde{B}^i_{jklm;s}y^s = -\bar{B}^i_{jklm}. \tag{9.38}$$

So far we have only used the condition $\mathbf{G}$ be R-quadratic.

From now on, we assume that $\mathbf{R}=0$. Then (9.35) simplifies to

$$\bar{B}^i_{jklm;s} = \bar{B}^i_{jkls;m}\ . \tag{9.39}$$

Contracting (9.39) with y^s and using (9.36), we obtain

$$\bar{B}^i_{jklm;s}y^s = \bar{B}^i_{jkls;m}y^s = [\bar{B}^i_{jkls}y^s]_{;m} = 0. \tag{9.40}$$

Proposition 9.3.2 *Let $(M,\mathbf{G})$ be a spray space with $\mathbf{R}=0$. For any parallel vector field $V(t)$ along a geodesic c, the following vector fields*

$$\begin{aligned}\bar{\mathbf{B}}(t) : &= \bar{\mathbf{B}}_{\dot{c}(t)}(V(t),V(t),V(t),V(t))\\ \tilde{\mathbf{B}}(t) : &= \tilde{\mathbf{B}}_{\dot{c}(t)}(V(t),V(t),V(t),V(t))\end{aligned}$$

must be in the following forms

$$\begin{aligned}\bar{\mathbf{B}}(t) &= E_1(t)\\ \tilde{\mathbf{B}}(t) &= -E_1(t)t + E_2(t),\end{aligned}$$

where $E_1(t)$ and $E_2(t)$ are two parallel vector fields along c.

Proof: Let $\dot{c}(t) = y^i(t)\frac{\partial}{\partial x^i}\Big|_{c(t)}$. It follows from (9.38) and (9.40)

$$\mathrm{D}_{\dot{c}}\bar{\mathbf{B}}(t) = \bar{B}^i_{jklm;s}y^s(t)V^j(t)V^k(t)V^l(t)V^m(t)\frac{\partial}{\partial x^i}\Big|_{c(t)} = 0, \tag{9.41}$$

$$\mathrm{D}_{\dot{c}}\tilde{\mathbf{B}}(t) = \tilde{B}^i_{jklm;s}y^s(t)V^j(t)V^k(t)V^l(t)V^m(t)\frac{\partial}{\partial x^i}\Big|_{c(t)} = -\bar{\mathbf{B}}(t). \tag{9.42}$$

(9.41) implies

$$\bar{\mathbf{B}}(t) = E_1(t)$$

is parallel along c. Take a parallel frame $\{P_i(t)\}_{i=1}^n$ along c with $E_1(t) = a_1P_1(t)$ for some constant a_1. Express

$$\tilde{\mathbf{B}}(t) = f^i(t)P_i(t).$$

Equation (9.42) implies

$$\frac{df^i}{dt}(t) = -a_1\delta_{1i}.$$

We obtain

$$f^1(t) = -a_1t + b_1, \qquad f^a(t) = b_a,$$

where $a = 2,\cdots,n$. Let

$$E_2(t) := b_iP_i(t).$$

We obtain

$$\tilde{\mathbf{B}}(t) = -E_1(t)t + E_2(t).$$

Q.E.D.

Example 9.3.1 Consider the following spray on a manifold M

$$\mathbf{G} := y^i \frac{\partial}{\partial x^i} - 2P(y)y^i \frac{\partial}{\partial y^i},$$

where P satisfies the homogeneity condition $P(\lambda y) = \lambda P(y), \forall \lambda > 0$. A direct computation yields

$$R^i_k = \Big(P^2 - y^j P_{x^j}\Big)\delta^i_k + \Big(2P_{x^k} - PP_{y^k} - y^j P_{x^j y^k}\Big)y^i. \tag{9.43}$$

We claim that $\mathbf{R} = 0$ if and only if P satisfies

$$P_{x^k} = PP_{y^k}. \tag{9.44}$$

First we assume that (9.44) holds. By the homogeneity of P, we obtain

$$y^j P_{y^j} = P, \qquad y^j P_{y^j y^k} = 0.$$

Thus

$$y^j P_{x^j} = y^j PP_{y^j} = P^2$$

$$\begin{aligned} 2P_{x^k} - PP_{y^k} - y^j P_{x^j y^k} &= 2PP_{y^k} - PP_{y^k} - y^j [PP_{y^j}]_{y^k} \\ &= PP_{y^k} - y^j P_{y^j} P_{y^k} - y^j PP_{y^j y^k} \\ &= PP_{y^k} - PP_{y^k} - 0 = 0. \end{aligned}$$

We conclude that $R^i_k = 0$. Now we assume that $R^i_k = 0$. It follows from (9.43) that

$$P^2 - y^j P_{x^j} = 0 \tag{9.45}$$

and

$$2P_{x^k} - PP_{y^k} - y^j P_{x^j y^k} = 0. \tag{9.46}$$

By (9.45), we obtain

$$y^j P_{x^j y^k} = [y^j P_{x^j}]_{y^k} - P_{x^k} = [P^2]_{y^k} - P_{x^k} = 2PP_{y^k} - P_{x^k}.$$

Plugging it into (9.46) yields

$$0 = 2P_{x^k} - PP_{y^k} - [2PP_{y^k} - P_{x^k}] = 3(P_{x^k} - PP_{y^k}).$$

Thus (9.44) holds. ♯

We see that (9.44) is same as (2.41). Thus there are lots of homogeneous functions satisfying (9.44). The above example leads to the following

Theorem 9.3.3 *There is a non-trivial R-flat spray on any strongly convex domain in* R^n*, whose geodesics are straight lines.*

There are many other R-flat sprays.

Example 9.3.2 Consider the following spray on R^3.

$$\mathbf{G} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} - 2a(x)u^2\frac{\partial}{\partial u} - 2b(y)v^2\frac{\partial}{\partial v} - 2c(z)w^2\frac{\partial}{\partial w}. \tag{9.47}$$

Clearly, $\mathbf{B} = 0$. According to Example 8.1.2, we know that $\mathbf{R} = 0$. Hence $\mathbf{G}$ is locally flat by Proposition 8.1.6. ♯

At the end, we take a look at the Berwald curvature of an isotropic spray. By definition, a spray is isotropic if in a standard local coordinate system (x^i, y^i), the coefficients $R^i_k(y)$ of the Riemann curvature are in the following form

$$R^i_k(y) = R(y)\delta^i_k + \tau_k(y)y^i, \tag{9.48}$$

where $\tau_k(y)$ satisfy

$$\tau_k(y)y^k = -R(y). \tag{9.49}$$

By (8.16) and (8.17), we obtain

$$\begin{aligned} R^i{}_{kl} &= \frac{1}{3}\Big\{R_{\cdot l}\delta^i_k - R_{\cdot k}\delta^i_l + \tau_k\delta^i_l - \tau_l\delta^i_k + (\tau_{k\cdot l} - \tau_{l\cdot k})y^i\Big\} \\ R_j{}^i{}_{kl} &= \frac{1}{3}\Big\{(\tau_{k\cdot j} - R_{\cdot k\cdot j})\delta^i_l - (\tau_{l\cdot j} - R_{\cdot l\cdot j})\delta^i_k \\ &\quad + (\tau_{k\cdot l} - \tau_{l\cdot k})\delta^i_j + (\tau_{k\cdot l\cdot j} - \tau_{l\cdot k\cdot j})y^i\Big\} \end{aligned}$$

Contracting (9.20) with y^k gives

$$B^i_{jml;k}y^k = R_j{}^i{}_{kl\cdot m}y^k = (R_j{}^i{}_{kl}y^k)_{\cdot m} - R_j{}^i{}_{ml}.$$

Observe that

$$\tau_{k\cdot l\cdot m}y^k = -R_{\cdot l\cdot m} - \tau_{l\cdot m} - \tau_{m\cdot l},$$

and

$$R_{\cdot j\cdot m\cdot l} + \tau_{k\cdot j\cdot m\cdot l}y^k = -\Big(\tau_{l\cdot j\cdot m} + \tau_{j\cdot l\cdot m} + \tau_{m\cdot l\cdot j}\Big).$$

By the above identities, we obtain

$$\begin{aligned} B^i_{jml;k}y^k &= \frac{1}{3}\Big\{\tau_{k\cdot j\cdot l}y^k\delta^i_m + \tau_{k\cdot j\cdot m}y^k\delta^i_l + \tau_{k\cdot l\cdot m}y^k\delta^i_j \\ &\quad + \Big(R_{\cdot j\cdot m\cdot l} + \tau_{k\cdot j\cdot m\cdot l}y^k\Big)y^i\Big\} \end{aligned} \tag{9.50}$$

By (9.50), we obtain a formula for the mean Berwald curvature coefficients $E_{jl} = \frac{1}{2}B^m_{jml}$

$$E_{jl;k}y^k = \frac{n+1}{6}\tau_{k\cdot j\cdot l}y^k. \tag{9.51}$$

The above identities will be used later.

Chapter 10

Structure Equations of Finsler Metrics

Sprays induced by Finsler metrics are much more special. In this case, we have not only the fundamental tensor, but also the Cartan torsion and the Landsberg curvature. In this chapter, we will study the relationship among these quantities.

10.1 Ricci Identities for g

Let (M, L) be Finsler space. In each fiber $\mathcal{V}TM_y$ of the vertical tangent bundle $\mathcal{V}TM = \mathrm{span}\Big\{\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^n}\Big\}$, define a bilinear symmetric form $\mathbf{g}_y$ by

$$\mathbf{g}_y\Big(\frac{\partial}{\partial y^i}|_y, \frac{\partial}{\partial y^j}|_y\Big) := g_{ij}(y), \tag{10.1}$$

where $g_{ij}(y) := \frac{1}{2}L_{y^iy^j}(y)$. We call $\mathbf{g} = \{\mathbf{g}_y\}_{y\in TM\setminus\{0\}}$ the *fundamental tensor*.

We can also define other important tensors on $\mathcal{V}TM$.

$$\begin{aligned}\mathbf{C}\Big(\frac{\partial}{\partial y^i}|_y, \frac{\partial}{\partial y^j}|_y, \frac{\partial}{\partial y^k}|_y\Big) &:= C_{ijk}(y),\\ \mathbf{L}\Big(\frac{\partial}{\partial y^i}|_y, \frac{\partial}{\partial y^j}|_y, \frac{\partial}{\partial y^k}|_y\Big) &:= L_{ijk}(y),\end{aligned}$$

where C_{ijk} and L_{ijk} are the coefficients of the Cartan torsion and the Landsberg curvature. We also call $\mathbf{C}$ and $\mathbf{L}$ the *Cartan torsion* and the *Landsberg curvature*, respectively. Similarly, we can define the Berwald curvature $\mathbf{B}$ as a tensor on $\mathcal{V}TM$.

Take a natural local coframe $\{\omega^i = dx^i\}$ for $\mathcal{H}^*TM$. Let

$$\omega_j^{\ i} = \Gamma^i_{jk}dx^k$$

denote the corresponding Berwald connection forms. Equation (6.30) is equivalent to

$$dg_{ij} - g_{kj}\omega_i^{\ k} - g_{ik}\omega_j^{\ k} = -2L_{ijk}\omega^k + 2C_{ijk}\omega^{n+k}. \qquad (10.2)$$

Define $g_{ij;k}$ and $g_{ij\cdot k}$ by

$$dg_{ij} - g_{kj}\omega_i^{\ k} - g_{ik}\omega_j^{\ k} = g_{ij;k}\omega^k + g_{ij\cdot k}\omega^{n+k}.$$

This gives

$$g_{ij;k} = -2L_{ijk}, \qquad g_{ij\cdot k} = 2C_{ijk}. \qquad (10.3)$$

Define $C_{ijk;l}$ and $C_{ijk\cdot l}$ by

$$dC_{ijk} - C_{pjk}\omega_i^{\ p} - C_{ipk}\omega_j^{\ p} - C_{ijp}\omega_k^{\ p} = C_{ijk;l}\omega^l + C_{ijk\cdot l}\omega^{n+l}. \qquad (10.4)$$

Similarly, we define $L_{ijk;l}$ and $L_{ijk\cdot l}$.

Differentiating (10.2) yields the following Ricci identities for $\mathbf{g}$.

$$\begin{aligned} -g_{pj}\Omega_i^{\ p} - g_{ip}\Omega_j^{\ p} &= 2L_{ijk;l}\omega^k \wedge \omega^l + 2L_{ijk\cdot l}\omega^k \wedge \omega^{n+l} \\ &\quad + 2C_{ijl;k}\omega^k \wedge \omega^{n+l} + 2C_{ijl\cdot k}\omega^{n+k} \wedge \omega^{n+l} + 2C_{ijp}\Omega^p. \qquad (10.5) \end{aligned}$$

It follows from (10.5) that

$$L_{ijk;l} - L_{ijl;k} = -\frac{1}{2}g_{pj}R_i^{\ p}{}_{kl} - \frac{1}{2}g_{ip}R_j^{\ p}{}_{kl} - C_{ijp}R^p{}_{kl}, \qquad (10.6)$$

$$C_{ijl;k} + L_{ijk\cdot l} = \frac{1}{2}g_{pj}B^p_{ikl} + \frac{1}{2}g_{ip}B^p_{jkl}, \qquad (10.7)$$

$$C_{ijk\cdot l} - C_{ijl\cdot k} = 0. \qquad (10.8)$$

Proof of Proposition 8.2.1: Contracting (10.6) with y^i and y^j yield

$$y^i g_{ip}R^p{}_{kl} = 0. \qquad (10.9)$$

Then contracting (10.6) with y^i and y^l, we obtain

$$0 = \frac{1}{2}g_{jp}R^p_k + \frac{1}{2}y^i g_{ip}R_j^{\ p}{}_{kl}y^l. \qquad (10.10)$$

Using (9.9), (10.9) and (10.10), we obtain

$$\begin{aligned} g_{jp}R^p_k &= -y^i g_{ip}R_j^{\ p}{}_{kl}y^l \\ &= y^i g_{ip}\Big(R_k^{\ p}{}_{lj} + R_l^{\ p}{}_{jk}\Big)y^l \\ &= -y^i g_{ip}R_k^{\ p}{}_{jl}y^l \\ &= g_{kp}R^p_j. \end{aligned}$$

This gives (8.57). Q.E.D.

We continue to derive other important identities. Recall (6.27)

$$L_{ijk} = -\frac{1}{2}y^s g_{sm} B^m_{ijk}.$$

Contracting (9.20) and (9.21) with $\frac{1}{2}y^s g_{is}$ also yield

$$L_{ijk;l} - L_{ijl;k} = \frac{1}{2}y^s g_{ps} R_i{}^p{}_{kl\cdot j}, \tag{10.11}$$

$$L_{ijk\cdot l} - L_{ijl\cdot k} = \frac{1}{2}g_{pk} B^p_{ijl} - \frac{1}{2}g_{pl} B^p_{ijk}. \tag{10.12}$$

Contracting (10.6) and (10.11) with y^l yield

$$L_{ijk;l}y^l = -C_{ijp}R^p_k - \frac{1}{2}g_{pj}R_i{}^p{}_{kl}y^l - \frac{1}{2}g_{ip}R_j{}^p{}_{kl}y^l, \tag{10.13}$$

$$L_{ijk;l}y^l = \frac{1}{2}y^s y^l g_{ps} R_i{}^p{}_{kl\cdot j}. \tag{10.14}$$

The right hand sides of (10.14) and (10.13) are equal. Contracting (10.7) with y^j yields

$$L_{jkl} = -\frac{1}{2}y^m g_{im} B^i_{jkl}\ . \tag{10.15}$$

This is our definition for L_{jkl} in (6.27). Contracting (10.7) with y^k yields

$$L_{ijk} = C_{ijk;l}y^l. \tag{10.16}$$

(10.16) is nothing but (6.31).

It follows from (10.7) that

$$g_{pk}B^p_{ijl} + g_{jp}B^p_{ikl} = 2C_{jkl;i} + 2L_{ijk\cdot l}, \tag{10.17}$$

$$g_{pi}B^p_{jkl} + g_{kp}B^p_{ijl} = 2C_{ikl;j} + 2L_{ijk\cdot l}. \tag{10.18}$$

(10.7) + (10.18) - (10.17) give rise to

$$B^p_{jkl} = g^{ip}\Big\{C_{ijl;k} + C_{ikl;j} - C_{jkl;i} + L_{ijk\cdot l}\Big\}. \tag{10.19}$$

Thus the horizontal covariant derivative of the Cartan torsion and the vertical covariant derivative of the Landsberg curvature determine the Berwald curvature.

Define

$$\bar{\mathbf{C}}_y(u,v,w,z) : = C_{ijk;l}(y)u^i v^j w^k z^l, \tag{10.20}$$

$$\tilde{\mathbf{C}}_y(u,v,w,z) : = C_{ijk\cdot l}(y)u^i v^j w^k z^l, \tag{10.21}$$

$$\bar{\mathbf{L}}_y(u,v,w,z) : = L_{ijk;l}(y)u^i v^j w^k z^l, \tag{10.22}$$

$$\tilde{\mathbf{L}}_y(u,v,w,z) : = L_{ijk\cdot l}(y)u^i v^j w^k z^l, \tag{10.23}$$

where $u = u^i \frac{\partial}{\partial x^i}|_x$, etc. In the following sections, we will show that these four quantities take very special forms along geodesics in a Finsler space of constant curvature.

We can also characterize Berwald metrics by $\bar{\mathbf{C}} = 0$. More precisely, we have the following

Lemma 10.1.1 *For a Finsler metric,* $\mathbf{B} = 0$ *if and only if* $\bar{\mathbf{C}} = 0$.

Proof: Suppose that $\mathbf{B} = 0$. By (10.15), $\mathbf{L} = 0$. Then (10.7) implies $\bar{\mathbf{C}} = 0$. Now suppose that $\bar{\mathbf{C}} = 0$. It follows from (10.16) that $\mathbf{L} = 0$. Then (10.19) implies that $\mathbf{B} = 0$. Q.E.D.

It follows from (10.3) that

$$g^{ij}{}_{;p} = 2g^{is}g^{jt}L_{stp}, \qquad g^{ij}{}_{.p} = -2g^{is}g^{jt}C_{stp}. \tag{10.24}$$

By (10.24) we obtain

$$g^{ij}C_{ijl;k} = \Big[g^{ij}C_{ijl}\Big]_{;k} - C_{ijl}g^{ij}{}_{;k} = I_{l;k} - 2C_{ijl}g^{is}g^{jt}L_{stk} \tag{10.25}$$

$$g^{ij}L_{ijk\cdot l} = \Big[g^{ij}L_{ijk}\Big]_{\cdot l} - L_{ijk}g^{ij}{}_{.l} = J_{k\cdot l} + 2L_{ijk}g^{is}g^{jt}C_{stl}, \tag{10.26}$$

where $I_i := g^{jk}C_{ijk}$ and $J_i := g^{jk}L_{ijk}$ denote the coefficients of the mean Cartan torsion and mean Landsberg curvature respectively.

From (10.19), (10.25) and (10.26) it follows that

$$E_{kl} = \frac{1}{2}g^{ij}\Big\{C_{ijl;k} + L_{ijk\cdot l}\Big\} = \frac{1}{2}\Big\{I_{l;k} + J_{k\cdot l}\Big\}. \tag{10.27}$$

Thus the mean Berwald curvature of a Finsler metric is determined by the horizontal covariant derivative of the mean Cartan torsion and the vertical derivative if the vertical covariant derivative of the mean Landsberg curvature.

Contracting (10.25) with y^k yields

$$I_{l;k}y^k = J_l. \tag{10.28}$$

Define

$$\bar{\mathbf{I}}_y(u,v) := I_{i;j}(y)u^iv^j, \tag{10.29}$$

where $u = u^i \frac{\partial}{\partial x^i}|_x$ and $v = v^j \frac{\partial}{\partial x^j}|_x$. We have the following

Lemma 10.1.2 *For a Finsler metric, if* $\bar{\mathbf{I}} = 0$, *then* $\mathbf{E} = 0$.

Proof: By assumption, $I_{l;k} = 0$. It follows from (10.28) that $J_l = 0$. Hence $J_{k\cdot l} = 0$. We conclude that $\mathbf{E} = 0$ by (10.27). Q.E.D.

Lemma 10.1.2 indicates that if the mean Cartan torsion is constant horizontally, then the mean Berwald curvature vanishes, and the Finsler metric is weakly Berwaldian.

10.2 Ricci Identities for C and L

In this section, we are going to derive some important identities for the second order covariant derivatives of the Cartan torsion and the Landsberg curvature. These identities are useful to study R-flat Finsler metrics and Finsler spaces of constant curvature.

Differentiating (10.4) yields

$$C_{pjk}\Omega_i^{\ p} + C_{ipk}\Omega_j^{\ p} + C_{ijp}\Omega_k^{\ p} = C_{ijk;l;m}\omega^l\wedge\omega^m$$
$$+C_{ijk;l\cdot m}\omega^l\wedge\omega^{n+m} - C_{ijk\cdot m;l}\omega^l\wedge\omega^{n+m} \qquad (10.30)$$
$$-C_{ijk\cdot m\cdot l}\omega^{n+l}\wedge\omega^{n+m} - C_{ijk\cdot p}\Omega^p. \qquad (10.31)$$

We obtain two Ricci identities

$$\begin{aligned} C_{ijk;l;m} - C_{ijk;m;l} &= C_{pjk}R_i^{\ p}{}_{lm} + C_{ipk}R_j^{\ p}{}_{lm} \\ &\quad + C_{ijp}R_k^{\ p}{}_{lm} + C_{ijk\cdot p}R^p{}_{lm}, & (10.32) \\ C_{ijk\cdot m;l} - C_{ijk;l\cdot m} &= C_{pjk}B^p_{ilm} + C_{ipk}B^p_{klm} + C_{ijp}B^p_{klm}, & (10.33) \\ C_{ijk\cdot m\cdot l} - C_{ijk\cdot l\cdot m} &= 0. & (10.34) \end{aligned}$$

Contracting (10.32) with y^m and using (10.16) yield

$$\begin{aligned} C_{ijk;l;m}y^m &= L_{ijk;l} + C_{pjk}R_i^{\ p}{}_l + C_{ipk}R_j^{\ p}{}_l \\ &\quad + C_{ijp}R_k^{\ p}{}_l + C_{ijk\cdot p}R^p_l, & (10.35) \end{aligned}$$

where $R_j^{\ i}{}_l := R_j^{\ i}{}_{lm}y^m$. Warning: $R_j^{\ i}{}_l$ are different from $R^i{}_{jl} := R_m^{\ i}{}_{jl}y^m$. Contracting (10.33) with y^l and using (10.16) yield

$$C_{ijk\cdot m;l}y^l = -C_{ijk;m} + L_{ijk\cdot m}. \qquad (10.36)$$

Similarly, we obtain the Ricci identities for the Landsberg curvature.

$$\begin{aligned} L_{ijk;l;m} - L_{ijk;m;l} &= L_{pjk}R_i^{\ p}{}_{lm} + L_{ipk}R_j^{\ p}{}_{lm} \\ &\quad + L_{ijp}R_k^{\ p}{}_{lm} + L_{ijk\cdot p}R^p_{lm}, & (10.37) \\ L_{ijk\cdot m;l} - L_{ijk;l\cdot m} &= L_{pjk}B^p_{ilm} + L_{ipk}B^p_{klm} + L_{ijp}B^p_{klm}, & (10.38) \\ L_{ijk\cdot m\cdot l} - L_{ijk\cdot l\cdot m} &= 0. & (10.39) \end{aligned}$$

Contracting (10.37) with y^m yields

$$\begin{aligned} L_{ijk;l;m}y^m &= [L_{ijk;m}y^m]_{;l} + L_{ijk\cdot p}R^p_l \\ &\quad + L_{pjk}R_i^{\ p}{}_l + L_{ipk}R_j^{\ p}{}_l + L_{ijp}R_k^{\ p}{}_l, & (10.40) \end{aligned}$$

where $R_k^{\ p}{}_l := R_k^{\ p}{}_{lm}y^m$. Contracting (10.38) with y^l yields

$$L_{ijk\cdot m;l}y^l = [L_{ijk;l}y^l]_{\cdot m} - L_{ijk;m}. \qquad (10.41)$$

10.3 R-Quadratic and R-Flat Finsler Metrics

In this section, we will study R-quadratic and R-flat Finsler metrics. R-flat Finsler metrics are much more special than R-flat sprays, especially on a compact manifold. We first study the case when the Finsler metric is R-quadratic, namely, $\mathbf{R}_y$ is quadratic in $y \in T_xM$ for all $x \in M$.

Proposition 10.3.1 *Let (M, L) be a Finsler space. Suppose that L is R-quadratic. Then for any geodesic $c(t)$ and any parallel vector field $V(t)$ along c, the following functions*

$$\mathbf{C}(t) := \mathbf{C}_{\dot c}(V(t), V(t), V(t)), \qquad \mathbf{L}(t) := \mathbf{L}_{\dot c}(V(t), V(t), V(t)) \tag{10.42}$$

must be in the following forms

$$\mathbf{L}(t) = \mathbf{L}(0), \tag{10.43}$$
$$\mathbf{C}(t) = \mathbf{L}(0)t + \mathbf{C}(0). \tag{10.44}$$

Proof: In local coordinates, let $\dot c(t) = y^i(t)\frac{\partial}{\partial x^i}$ and $V(t) = V^i(t)\frac{\partial}{\partial x^i}$.

$$\mathbf{L}'(t) = L_{ijk;m}y^m(t)V^j(t)V^k(t)V^l(t). \tag{10.45}$$

By assumption, $R_j{}^i{}_{kl\cdot m} = 0$. It follows from (10.14) and (10.16) that

$$L_{ijk;l}y^l = 0, \tag{10.46}$$
$$C_{ijk;l}y^l = L_{ijk}. \tag{10.47}$$

By (10.46) and (10.47), we obtain

$$\begin{aligned}\mathbf{L}'(t) &= L_{ijk;l}y^l(t)V^i(t)V^j(t)V^k(t) = 0,\\ \mathbf{C}'(t) &= C_{ijk;l}y^l(t)V^i(t)V^j(t)V^k(t)\\ &= L_{ijk}V^i(t)V^j(t)V^k(t) = \mathbf{L}(t).\end{aligned}$$

Then (10.43) and (10.44) follow. Q.E.D.

Theorem 10.3.2 *Let (M, F) be a positive definite R-quadratic Finsler space. Suppose that M is compact or F is positively complete with bounded Cartan torsion. Then F must be a Landsberg metric and the Cartan torsion is constant along any geodesic.*

Proof: For an arbitrary unit vector $y \in T_xM$ and an arbitrary vector $v \in T_xM$, let $c(t)$ be the geodesic with $\dot c(0) = y$ and $V(t)$ the parallel vector field along c with $V(0) = v$. Define $\mathbf{C}(t)$ and $\mathbf{L}(t)$ as in (10.42). Then

$$\mathbf{C}(t) = \mathbf{L}(0)t + \mathbf{C}(0).$$

Suppose that $\mathbf{C}$ is bounded, i.e.,

$$\|\mathbf{C}\| := \sup_{y\in TM} \frac{|\mathbf{C}_y(v,v,v)|}{[g_y(v,v)]^{\frac{3}{2}}} < \infty.$$

By Lemma 7.3.2,

$$Q := g_{\dot{c}(t)}(V(t),V(t))$$

is a positive constant. Thus

$$|\mathbf{C}(t)| \leq \|\mathbf{C}\| Q^{\frac{3}{2}} < \infty.$$

and $\mathbf{C}(t)$ is a bounded function on $[0,\infty)$. By (10.44), we conclude that

$$\mathbf{L}_y(v,v,v) = \mathbf{L}(0) = 0.$$

Therefore $\mathbf{L}=0$ and F is a Landsberg metric. Q.E.D.

The reader is referred to [BaMa5] and [BaMa6] for some results on the local geometric structure of R-quadratic Finsler metrics. In the above mentioned papers, the authors do not assume that the Finsler metrics are positive definite.

Now it is clear that for a positive definite Finsler metric F on a compact manifold M, if F is a Berwald metric, then it is R-quadratic; if F is R-quadratic, then it is a Landsberg metric. An open problem in Finsler geometry is whether every Landsberg metric is a Berwald metric.

Corollary 10.3.3 *For any positively complete Randers metric $F=\alpha+\beta$ on a manifold M, if F is R-quadratic, then it must be a Berwald space.*

Proof: First we know that the Cartan torsion of F must be bounded. In fact,

$$\|\mathbf{C}\| \leq \frac{3}{\sqrt{2}}.$$

By Theorem 10.3.2, F is a Landsberg metric. Hence F is a Berwald metric by [Ma2] [HaIc1] [SSAY] [Ki]. Q.E.D.

In [BaMa6], Bácsó and Matsumoto classify R-quadratic Randers metrics. Their results indicate that there might be local R-quadratic Randers metrics which are not of Berwald type.

For a submanifold M in a positive definite Minkowski space (V,F), the Cartan torsion is always bounded. Then we obtain the following

Corollary 10.3.4 *For any positively complete submanifold M in a positive definite Minkowski space (V,F), if the induced Finsler metric is R-quadratic, then it must be a Landsberg space.*

According to Corollary 10.3.4, any positively complete R-flat submanifold in a Minkowski space is a Landsberg space. We will prove that it must be locally Minkowskian space. See Corollary 10.3.8.

Now we discuss R-flat Finsler metrics. According to Theorem 9.3.3, there is a non-trivial R-flat spray $\mathbf{G}$ on any strongly convex domain Ω in R^n, whose geodesics are straight lines. Since R-flat sprays are isotropic, by a theorem of Grifone-Muzsnay [GrMu], we conclude that $\mathbf{G}$ is locally Finslerian. Namely $\mathbf{G}$ is locally induced by a Finsler metric L. Therefore, we have the following

Theorem 10.3.5 *There is a non-trivial R-flat Finsler metric in an open subset in* R^n*, whose geodesics are straight lines.*

The spray of the Finsler metric in Theorem 10.3.5 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2F(y)y^i \frac{\partial}{\partial y^i} \tag{10.48}$$

where F is the Funk metric on a strongly convex domain Ω in R^n. When $\Omega = \mathrm{B}^n$ is the standard unit ball, the Funk metric F is given by

$$F(y) = \frac{\sqrt{|y|^2 - \Big(|x|^2|y|^2 - \langle x, y\rangle^2\Big)}}{1-|x|^2} + \frac{\langle x, y\rangle}{1-|x|^2}, \qquad y \in T_x\mathrm{B}^n = \mathrm{R}^n,$$

where $|\cdot|$ and $\langle , \rangle$ denote the standard Euclidean norm and inner product respectively. Is there a Finsler metric on the whole unit ball B^n that induces the above spray in (10.48) ? If any, can it be expressed by elementary functions ?

Next, we are going to study general properties of R-flat Finsler metrics. Recall the definitions of $\bar{\mathbf{C}}$, $\tilde{\mathbf{C}}$, $\bar{\mathbf{L}}$ and $\tilde{\mathbf{L}}$ in (10.20)-(10.23). $\tilde{\mathbf{C}}_y$ is symmetric. But $\bar{\mathbf{C}}_y$, $\bar{\mathbf{L}}_y$ and $\tilde{\mathbf{L}}_y$ are not symmetric. Nevertheless, $\bar{\mathbf{C}}_y(u, v, w, z)$ is completely determined by $\bar{\mathbf{C}}_y(v, v, v, w)$. The same is true for $\bar{\mathbf{L}}_y$ and $\tilde{\mathbf{L}}_y$.

Let c be a geodesic and $V(t), W(t)$ be parallel vector fields along c. Set

$$\bar{\mathbf{C}}(t) := \bar{\mathbf{C}}_{\dot{c}(t)}(V(t), V(t), V(t), W(t)), \tag{10.49}$$
$$\tilde{\mathbf{C}}(t) := \tilde{\mathbf{C}}_{\dot{c}(t)}(V(t), V(t), V(t), W(t)), \tag{10.50}$$
$$\bar{\mathbf{L}}(t) := \bar{\mathbf{L}}_{\dot{c}(t)}(V(t), V(t), V(t), W(t)), \tag{10.51}$$
$$\tilde{\mathbf{L}}(t) := \tilde{\mathbf{L}}_{\dot{c}(t)}(V(t), V(t), V(t), W(t)). \tag{10.52}$$

Proposition 10.3.6 *Let* (M, L) *be a R-flat Finsler space. For any geodesic* c *and any parallel vector fields* $V(t)$ *and* $W(t)$ *along* c*, the above defined functions* $\bar{\mathbf{C}}(t)$*,* $\tilde{\mathbf{C}}(t)$*,* $\bar{\mathbf{L}}(t)$ *and* $\tilde{\mathbf{L}}(t)$ *satisfy*

$$\bar{\mathbf{C}}(t) = \bar{\mathbf{L}}(0)t + \bar{\mathbf{C}}(0), \tag{10.53}$$
$$\tilde{\mathbf{C}}(t) = -\bar{\mathbf{L}}(0)t^2 + (\tilde{\mathbf{L}}(0) - \bar{\mathbf{C}}(0))t + \tilde{\mathbf{C}}(0), \tag{10.54}$$
$$\bar{\mathbf{L}}(t) = \bar{\mathbf{L}}(0), \tag{10.55}$$
$$\tilde{\mathbf{L}}(t) = -\bar{\mathbf{L}}(0)t + \tilde{\mathbf{L}}(0). \tag{10.56}$$

Proof: It follows from (10.35) and (10.36) that

$$\begin{aligned} C_{ijk;l;m}y^m &= L_{ijk;l}, \\ C_{ijk\cdot m;l}y^l &= -C_{ijk;m} + L_{ijk\cdot m}. \end{aligned}$$

Since $\mathbf{R} = 0$, by (10.14), we know that $L_{ijk;l}y^l = 0$. Thus (10.40) and (10.41) simplify to

$$\begin{aligned} L_{ijk;l;m}y^m &= 0, \\ L_{ijk\cdot l;m}y^m &= -L_{ijk;l}. \end{aligned}$$

The above four identities imply

$$\begin{aligned} \bar{\mathbf{C}}'(t) &= \bar{\mathbf{L}}(t), \\ \tilde{\mathbf{C}}'(t) &= -\bar{\mathbf{C}}(t) + \tilde{\mathbf{L}}(t), \\ \bar{\mathbf{L}}'(t) &= 0, \\ \tilde{\mathbf{L}}'(t) &= -\bar{\mathbf{L}}(t). \end{aligned}$$

Solving the above system of ODEs, we obtain (10.53)-(10.56). Q.E.D.

By Propositions 10.3.1 and 10.3.6, we immediately obtain the following important theorem which slightly generalizes a result by Akbar-Zedah [AZ2]. Compare Theorem 10.3.2.

Theorem 10.3.7 *Let (M, F) be a positive definite R-flat Finsler space. If F is positively complete and satisfies the following two conditions*

(i) $\mathbf{C}$ *is bounded or* $\mathbf{L} = 0$,

(ii) $\tilde{\mathbf{C}}$ *is bounded,*

then it must be locally Minkowskian.

Proof: Take an arbitrary geodesic $c(t)$ and a parallel vector field $V(t)$ along c. Define $\bar{\mathbf{C}}(t)$, $\tilde{\mathbf{C}}(t)$, $\bar{\mathbf{L}}(t)$ and $\tilde{\mathbf{L}}(t)$ as above. Suppose $\mathbf{C}$ is bounded. First, by Theorem 10.3.2, we have $\mathbf{L}(0) = 0$. (10.54) simplifies to

$$\tilde{\mathbf{C}}(t) = -\bar{\mathbf{C}}(0)t + \tilde{\mathbf{C}}(0).$$

Since $\tilde{\mathbf{C}}$ is bounded, we conclude that

$$\bar{\mathbf{C}}(0) = 0.$$

Since both $c(t)$ and $V(t)$ are arbitrary, we conclude that $\bar{\mathbf{C}} = 0$ everywhere. Therefore, F is a Berwald metric by Lemma 10.1.1. By Proposition 8.2.4, F is locally Minkowskian space. Q.E.D.

Corollary 10.3.8 *Let* (V, F) *be a positive definite Minkowski space and* $\varphi : M \to (\mathrm{V}, F)$ *be a positively complete submanifold. If the induced metric* $\bar{F} = \varphi^* F$ *is R-flat, then* M *must be locally Minkowskian.*

Proof: It suffices to prove that both $\mathbf{C}$ and $\tilde{\mathbf{C}}$ are bounded. This is left to the readers. Compare [Sh4]. Q.E.D.

Example 10.3.1 ([SSAY][YaSh]) Consider a Randers metric $F = \alpha + \beta$, where α is a Riemannian metric and β is a 1-form with $\|\beta\| := \sup |\beta(y)|/\alpha(y) < 1$. Yasuda-Shimada proved that F is R-flat if and only if it is locally Minkowskian. In this case, α is R-flat and β is parallel with respect to α. Thus R-flat Randers metrics are trivial[1] ♯

[1] This result is fale without additional assumption that the Randers metric is positively complete. There is a non-trivial incomplete R-flat Randers metric.

Chapter 11

Finsler Spaces of Scalar Curvature

By Definition 8.2.3, a Finsler metric is of scalar curvature if the Riemann curvature is in the following form

$$\mathbf{R}_y(u) = \lambda(y)\Big\{ g_y(y,y)\, u - g_y(y,u)\, y \Big\}. \tag{11.1}$$

According to Lemma 8.2.2, a Finsler metric is of scalar curvature if and only if the induced spray is isotropic. Thus a Finsler metric of scalar curvature may be also called an *isotropic Finsler metric*. In this book, we will still use the traditional terminology.

There are many complete and incomplete Finsler metrics of scalar curvature. In dimension $n \geq 3$, a Riemannian metric is of scalar curvature if and only if it is of constant curvature. See Proposition 11.1.1 below. For a Finsler metric on an open subset $\Omega \subset \mathrm{R}^n$, if the geodesics are straight lines, then it must be of scalar curvature. The Funk metrics in (2.40) and the Klein metric in (2.45) are actually of negative constant curvature. The spherical metrics in (2.32) and (2.36) are of positive constant curvature $\mathbf{K} = 1$.

In this chapter, we will discuss some basic properties of Finsler metrics of scalar curvature.

11.1 Finsler Metric of Scalar Curvatures

Let (M, L) be a Finsler space. For a flag $\{P, y\}$ in T_xM, where $P \subset T_xM$ is a tangent plane containing y, the flag curvature $\mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(P,y) := \frac{g_y(\mathbf{R}_y(u), u)}{g_y(y,y)g_y(u,u) - g_y(y,u)g_y(y,u)}, \tag{11.2}$$

where $u \in P$ such that $P = \mathrm{span}\{u, y\}$. We know that F is of scalar curvature if and only if there is a scalar function $\lambda(y)$ on TM such that for any $y \in T_xM \setminus \{0\}$

and any tangent plane $P \subset T_xM$ containing y,

$$\mathbf{K}(P, y) = \lambda(y).$$

By Lemmas 8.1.10 and 8.2.2, we know that every two-dimensional Finsler metric is of scalar curvature $\lambda(y)$ and the Gauss curvature $\mathbf{K}(y) = \lambda(y)$. When the Finsler metric is Riemannian, the Gauss curvature is a scalar function on M. This fact is left for the reader to verify.

Example 11.1.1 Consider a pseudo-Riemannian metric L on an open subset $\mathcal{U} \subset \mathrm{R}^2$,

$$L := -a(x,y)u^2 + c(x,y)v^2, \tag{11.3}$$

where $a(x,y)$ and $c(x,y)$ are positive C^∞ functions on $\mathcal{U}$. An easy computation gives

$$\mathbf{K} = \frac{1}{2\sqrt{ac}}\Big[\Big(\frac{c_x}{\sqrt{ac}}\Big)_x - \Big(\frac{a_y}{\sqrt{ac}}\Big)_y\Big]. \tag{11.4}$$

Note that $\mathbf{K}$ is a scalar function on $\mathcal{U}$. ♯

Let $\varphi : (a,b)\times \mathrm{S}^1 \to \mathrm{R}^3$ be a surface of revolution around the third coordinate line in R^3

$$\varphi(x,y) := \Big(f(x)\cos y,\ f(x)\sin y,\ x\Big).$$

R^3 is equipped with the following pseudo-Euclidean metric

$$g = u^2 + v^2 - w^2.$$

The induced Finsler metric L on $M^2 := (a,b)\times \mathrm{S}^1$ under φ is given by

$$L = -\Big(1 - f'(x)^2\Big)u^2 + f(x)^2 v^2.$$

If $|f'(x)| < 1$, L is a pseudo-Riemannian metric. If $|f'(x)| > 1$, L is a Riemannian metric. In either case, the Gauss curvature $\mathbf{K}$ at $p = (x,y)$ is given by

$$\mathbf{K} = \frac{f''(x)}{f(x)\Big[1 - f'(x)^2\Big]^2}. \tag{11.5}$$

(i) Take

$$f(x) = \sqrt{1 + x^2}.$$

The resulting metric has Gauss curvature $\mathbf{K} = 1$.

(ii) Take

$$f(x) = ax + b, \qquad a \neq \pm 1.$$

The resulting metric has Gauss curvature $\mathbf{K} = 0$. Note that when $a = \pm 1$, the induced metric on M^2 is degenerate.

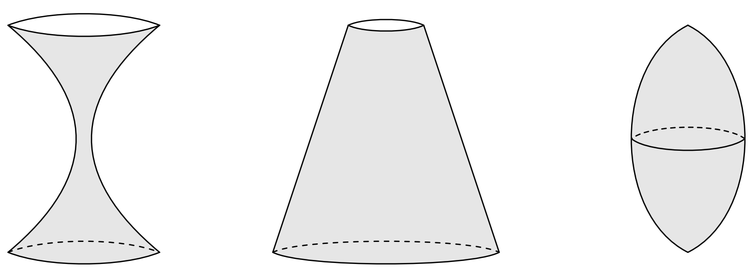
(iii) Take a function f satisfying

$$f''(x) + f(x)\Big[1 - f'(x)^2\Big]^2 = 0, \tag{11.6}$$

with

$$f(0) = 1, \qquad f'(0) \neq \pm 1.$$

The resulting metric has Gauss curvature $\mathbf{K} = -1$. The above surfaces in R^3 look like



$\mathbf{K} = 1$ $\qquad$ $\mathbf{K} = 0$ $\qquad$ $\mathbf{K} = -1$

We see that a surface of constant curvature $\mathbf{K} = 1$ in (R^3, g) is curved as a surface of curvature $\mathbf{K} = -1$ in the Euclidean space $\mathbb{R}^3$. Compare Example 11.3.3 below.

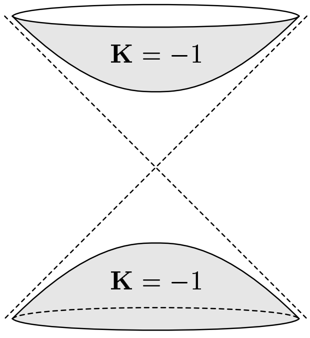
Let

$$f(x) = \sqrt{x^2 - 1}, \qquad |x| \geq 1. \tag{11.7}$$

One can easily verify that $f(x)$ is a solution of (11.6). The induced metric L on the surface is given by

$$L = \frac{1}{x^2 - 1} u^2 + (x^2 - 1) v^2.$$

L also has Gauss curvature $\mathbf{K} = -1$. This surface in R^3 looks like



This hyperbolic surface was discovered by Beltrami in 1868.

Example 11.1.2 Consider a Randers metric $F = \alpha + \beta$ on a manifold M, where α is a Riemannian metric and β is a close 1-form. According to Example 8.2.1, F is of scalar curvature if and only if α is of scalar curvature (constant curvature when $n = \dim M \geq 3$). ♯

Assume that a Finsler metric L is of scalar curvature $\lambda = \lambda(y)$. So the Riemann curvature is in the form (11.1). In local coordinates, (11.1) is expressed as follows

$$R^i_k(y) = \lambda(y)\Big\{ L(y)\delta^i_k - g_{jk}(y) y^j y^i \Big\}. \tag{11.8}$$

For simplicity, let

$$\lambda_{\cdot i} := \lambda_{y^i}, \qquad \lambda_{\cdot i \cdot j} := \lambda_{y^i y^j}, \qquad \cdots,$$

and

$$L_{\cdot i} := L_{y^i} = 2 g_{is} y^s.$$

Let

$$h_{ij} := g_{ij} - \frac{1}{4L} L_{\cdot i} L_{\cdot j} = g_{ij} - \frac{1}{L} g_{is} y^s \, g_{jt} y^t, \tag{11.9}$$

and

$$h^i_j := g^{ip} h_{jp} = \delta^i_j - \frac{1}{2L} L_{\cdot j} y^i = \delta^i_j - \frac{1}{L} g_{js} y^s \, y^i.$$

We are going to apply Bianchi identities (9.19)-(9.23) to study a Finsler metrics of scalar curvature $\lambda = \lambda(y)$. We show that in dimension ≥ 3, if, in addition, $\lambda = \lambda(x)$ is a scalar function on M, then λ must be a constant.

Plugging (11.8) into (8.17) gives

$$\begin{aligned} R_j{}^i{}_{kl} \;=\;& \frac{\lambda_{\cdot j \cdot l}}{3} L h^i_k - \frac{\lambda_{\cdot j \cdot k}}{3} L h^i_l + \frac{\lambda_{\cdot j}}{2}\Big\{ L_{\cdot l}\delta^i_k - L_{\cdot k}\delta^i_l \Big\} \\ & + \frac{\lambda_{\cdot l}}{3}\Big\{ L_{\cdot j}\delta^i_k - g_{jk} y^i - \frac{1}{2} L_{\cdot k}\delta^i_j \Big\} - \frac{\lambda_{\cdot k}}{3}\Big\{ L_{\cdot j}\delta^i_l - g_{jl} y^i - \frac{1}{2} L_{\cdot l}\delta^i_j \Big\} \\ & + \lambda\Big\{ g_{jl}\delta^i_k - g_{jk}\delta^i_l \Big\} \end{aligned} \tag{11.10}$$

Contracting (11.10) with y^j yields

$$R^i{}_{kl} = \frac{\lambda_{\cdot l}}{3} L\, h^i_k - \frac{\lambda_{\cdot k}}{3} L\, h^i_l + \frac{1}{2}\lambda\Big\{ L_{\cdot l}\delta^i_k - L_{\cdot k}\delta^i_l \Big\}. \tag{11.11}$$

Differentiating (11.11) along the direction $y^m \frac{\delta}{\delta x^m}$ yields

$$R^i{}_{kl;m}y^m = \frac{\lambda_{\cdot l;m}y^m}{3} L\, h^i_k - \frac{\lambda_{k;m}y^m}{3} L\, h^i_l + \frac{1}{2}\lambda_{;m}y^m\Big\{ L_{\cdot l}\delta^i_k - L_{\cdot k}\delta^i_l \Big\}. \tag{11.12}$$

Differentiating (11.8) along the direction $\frac{\delta}{\delta x^l}$ yields

$$R^i_{k;l} = \lambda_{;l} L\, h^i_k. \tag{11.13}$$

Plugging (11.12) and (11.13) into (9.23), we obtain

$$\begin{aligned} \lambda_{;l} L\, h^i_k - \lambda_{;k} L\, h^i_l - \frac{1}{2}\lambda_{;m}y^m\Big\{ L_{\cdot l}\delta^i_k - L_{\cdot k}\delta^i_l \Big\} \\ = \frac{\lambda_{\cdot l;m}y^m}{3} L\, h^i_k - \frac{\lambda_{k;m}y^m}{3} L\, h^i_l. \end{aligned} \tag{11.14}$$

Now we assume that $\lambda = \lambda(x)$ is a function of x only. Then $\lambda_{\cdot i} = 0$ and (11.14) simplifies to

$$\lambda_{;l} L\, h^i_k - \lambda_{;k} L\, h^i_l - \frac{1}{2}\lambda_{;m}y^m\Big\{ L_{\cdot l}\delta^i_k - L_{\cdot k}\delta^i_l \Big\} = 0. \tag{11.15}$$

Suppose that $n = \dim M \geq 3$. Fix a vector $y \in T_xM$ with $L(y) \neq 0$. There are two non-parallel vectors $u, v \in T_xM$ which are g_y-orthogonal to y, i.e.,

$$L_{\cdot k}u^k = 0 = L_{\cdot k}v^k. \tag{11.16}$$

Contracting (11.15) with u^k and v^l yields

$$(v^l\lambda_{;l}L)\, u^i = (u^k\lambda_{;k}L)\, v^i.$$

We conclude that for $u \in T_xM$ with $g_y(y, u) = 0$, the following holds

$$d\lambda(u) = \lambda_{\cdot k}u^k = 0. \tag{11.17}$$

By the continuity of λ, we conclude that (11.17) holds for any $u \in T_xM$. This proves the following

Proposition 11.1.1 *Let (M, L) be a Finsler space of dimension $n \geq 3$. If L is of scalar curvature $\lambda = \lambda(x)$ depending on position only, then $\lambda(x) = constant$.*

Next, we are going to give a brief argument on Numata's theorem that any Landsberg metric of scalar curvature in dimension ≥ 3 must be Riemannian on the open subset of M where the curvature scalar $\lambda(y) \neq 0$.

Contracting (11.10) with y^l yields

$$\begin{aligned} R_j{}^i{}_{kl} y^l &= \frac{2}{3}\lambda_{.j}\Big\{ L\delta^i_k - \frac{1}{2}L_{.k}y^i \Big\} + \frac{1}{3}\lambda_{.k}\Big\{ L\delta^i_j - \frac{1}{2}L_{.j}y^i \Big\} \\ &\quad + \lambda \Big\{ \frac{1}{2}L_{.j}\delta^i_k - g_{jk}y^i \Big\}. \end{aligned} \tag{11.18}$$

Plugging (11.18) into (10.13), one obtains

$$L_{ijk;l}y^l = -\frac{\lambda_{.i}}{3}L\ h_{jk} - \frac{\lambda_{.j}}{3}L\ h_{ik} - \frac{\lambda_{.k}}{3}L\ h_{ij} - \lambda L C_{ijk}. \tag{11.19}$$

From now on, we assume that L is a Landsberg metric. (11.19) reduces to

$$C_{ijk} = -\frac{1}{3\lambda}\Big\{ \lambda_{.i}h_{jk} + \lambda_{.j}h_{ik} + \lambda_{.k}h_{ij} \Big\}. \tag{11.20}$$

Contracting (11.20) with g^{jk} yields

$$C_i = -\frac{n+1}{3\lambda}\lambda_{.i}, \tag{11.21}$$

where $C_i := g^{jk}C_{ijk}$. Substituting (11.21) back to (11.20), we obtain

$$C_{ijk} = \frac{1}{n+1}\Big\{ C_i h_{jk} + C_j h_{ik} + C_k h_{ij} \Big\}. \tag{11.22}$$

Thus L is C-reducible in the sense of Matsumoto [Ma1]. In other words, any Landsberg metric of scalar curvature $\lambda(y) \neq 0$ must be C-reducible. When $n \geq 3$, L is actually Riemannian.

Proposition 11.1.2 ([Nu]) *Let (M, L) be a Landsberg space of dimension ≥ 3. Suppose that L is of scalar curvature $\lambda \neq 0$. Then L is Riemannian.*

S. Numata first proved the proposition for Berwald spaces of scalar curvature. Then he proved the proposition for Landsberg space of scalar curvature based on a result of M. Matsumoto on C-reducible spaces [Ma1].

What happens when $\mathbf{R} = 0$ for a Finsler metric L in Proposition 11.1.2 ? According to Theorem 10.3.7, if, in addition, L is positively complete and positive definite, then it is locally Minkowskian, provided that $\tilde{\mathbf{C}}$ is bounded. Finsler metrics constructed based on Proposition 8.2.7 or Theorem 10.3.5 are in general R-flat non-Landsbergian metrics. A natural question is whether or not there is a Landsberg metric on R^n with $\mathbf{R} = 0$, which is not locally Minkowskian.

Note that all two-dimensional Finsler metrics are of scalar curvature. In dimension two, Numata's theorem might not be true. Assume that L a two-dimensional Berwald metric. Then the Gauss curvature $\mathbf{K}$ satisfies a special equation along the indicatrix at each point. If L is positive definite with $\mathbf{K} \neq 0$, one can see that the main scalar must be zero. Thus, L is Riemannian. This fact is due to Z. Szabó [Sz1]. Refer to Proposition 6.1.6. See also [BaChSh1] for a proof. Note that if $\mathbf{K} = 0$, then L is locally Minkowskian.

Anyway, we still do not know whether or not there is a (two-dimensional) Landsberg metric which is not Berwaldian ! Is there a non-trivial two-dimensional R-flat Landsberg metric? We do not have any clue yet.

Let L be a Finsler metric of scalar curvature $\lambda(y)$. Differentiating (11.10) with respect to y^m gives a formula for $R_j{}^i{}_{kl\cdot m}$ expressed in terms of λ and its derivatives. Contracting $R_j{}^i{}_{kl\cdot m} = B^i_{jml;k} - B^i_{jkm;l}$ with y^k, one obtains

$$\begin{aligned} B^i_{jml;k}y^k &= 2\lambda C_{jlm}y^i && (11.23)\\ &\quad -\frac{\lambda_{\cdot j}}{3}\Big\{\frac{1}{2}L_{\cdot l}\delta^i_m + \frac{1}{2}L_{\cdot m}\delta^i_l - 2g_{lm}y^i\Big\}\\ &\quad -\frac{\lambda_{\cdot l}}{3}\Big\{\frac{1}{2}L_{\cdot j}\delta^i_m + \frac{1}{2}L_{\cdot m}\delta^i_j - 2g_{jm}y^i\Big\}\\ &\quad -\frac{\lambda_{\cdot m}}{3}\Big\{\frac{1}{2}L_{\cdot j}\delta^i_l + \frac{1}{2}L_{\cdot l}\delta^i_j - 2g_{jl}y^i\Big\}\\ &\quad -\frac{\lambda_{\cdot j\cdot m}}{3}L\, h^i_l - \frac{\lambda_{\cdot j\cdot l}}{3}L\, h^i_m - \frac{\lambda_{\cdot l\cdot m}}{3}L\, h^i_j, && (11.24)\end{aligned}$$

where $h^i_j = \delta^i_j - \frac{1}{2L}L_{\cdot j}y^i$. It follows from (11.24) that

$$E_{jl;k}y^k = -\frac{n+1}{6}\Big\{\frac{1}{2}\lambda_{\cdot j}L_{\cdot l} + \frac{1}{2}\lambda_{\cdot l}L_{\cdot j} + \lambda_{\cdot j\cdot l}L\Big\}. \qquad (11.25)$$

One can also derive (11.24) and (11.25) from (9.50) and (9.51) respectively by taking $R = \lambda L$ and $\tau_k = -\frac{1}{2}\lambda L_{\cdot k}$.

Assume that L is of scalar curvature $\lambda = \lambda(x)$ depending on x only (Note: $\lambda = \mathit{constant}$ when dim > 2 by Proposition 11.1.1). In this case, (11.19), (11.24) and (11.25) simplify to

$$\begin{aligned} L_{ijk;m}y^m &= -\lambda L C_{ijk}, && (11.26)\\ E_{ij;m}y^m &= 0, && (11.27)\\ B^i_{jkl;m}y^m &= 2\lambda C_{jkl}y^i. && (11.28)\end{aligned}$$

The above identities will be used in the following section.

11.2 Finsler Metrics of Constant Curvature

The local structures of Finsler metrics of constant curvature have not been completely understood. The Funk metric F on a strongly convex domain $\Omega \subset \mathrm{R}^n$ is non-reversible with constant curvature $\mathbf{K} = -1/4$. The Funk metric F is positively complete, while its reserve $\bar{F}(y) := F(-y)$ is negatively complete. The Klein metric $\tilde{F} := \frac{1}{2}(\bar{F} + F)$ is reversible and complete with constant curvature $\mathbf{K} = 1$. The Bryant metrics on S^n are non-reversible with constant curvature $\mathbf{K} = 1$. All of the above mentioned Finsler metrics are locally projectively flat. Namely, at every point, there is a local coordinate system in which geodesics are straight lines. See Chapter 13 for more discussion on the projective geometry of Finsler spaces.

Let us take a look at a Finsler metric $L = L(x, y, u, v)$ on an open subset $\mathcal{U} \subset \mathrm{R}^2$ in the form

$$L = \Big[u\phi\Big(x, y, \frac{v}{u}\Big)\Big]^2$$

where $\phi = \phi(x, y, \xi)$ is a function on $\mathcal{U} \times \mathrm{R}$ satisfying $\phi_{\xi\xi} \neq 0$. According to Proposition 8.2.7, if ϕ satisfies (8.83), i.e.,

$$\phi_y = 0, \qquad \frac{\phi_x}{\phi^2} + \Big(\frac{\phi_{x\xi}}{\phi\phi_{\xi\xi}}\Big)_\xi = 0, \tag{11.29}$$

then the Gauss curvature $\mathbf{K} = 0$. For a solution ϕ of (11.29), if

$$\Phi := -\frac{\phi_{x\xi}}{\phi_{\xi\xi}}$$

is not a polynomial of degree three or less in ξ, then L is not projectively flat. There are many solutions ϕ satisfying the above conditions. Thus there are many non-projectively flat R-flat Finsler metrics. But it is not clear whether or not the resulting Finsler metrics are positive definite and regular in all directions. See Remark 13.6.7 below.

Recently, Bao and Shen have constructed a family of Randers metrics on S^3 of constant curvature $\mathbf{K} = 1$. These metrics are not locally projectively flat. The details are given in [BaSh2] and Example 11.2.1 below.

The classification of Finsler metrics of constant curvature is far from being done. In this section, we are going to study the Cartan torsion, the Landsberg curvature and the Berwald curvature along geodesics in a Finsler space of constant curvature λ. We will also discuss some interesting examples.

For a number $\lambda \in \mathrm{R}$, let

$$\mathbf{s}_\lambda(t) : \;= \begin{cases} \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}} & \text{if } \lambda > 0 \\ t & \text{if } \lambda = 0 \\ \frac{\sinh(\sqrt{-\lambda}t)}{\sqrt{-\lambda}} & \text{if } \lambda < 0. \end{cases} \tag{11.30}$$

$$\mathbf{c}_\lambda(t) := \begin{cases} \cos(\sqrt{\lambda}t) & \text{if } \lambda > 0 \\ 1 & \text{if } \lambda = 0 \\ \cosh(\sqrt{-\lambda}t) & \text{if } \lambda < 0. \end{cases} \tag{11.31}$$

Note that

$$\mathbf{s}'_\lambda(t) = \mathbf{c}_\lambda(t), \qquad \mathbf{c}'_\lambda(t) = -\lambda \mathbf{s}_\lambda(t).$$

Thus both $\mathbf{s}_\lambda(t)$ and $\mathbf{c}_\lambda(t)$ satisfy the following equation

$$y''(t) + \lambda y(t) = 0.$$

Let (M, L) be a Finsler space. Take a geodesic $c(t)$. By Lemma 4.2.2,

$$\frac{d}{dt}[L(\dot{c})] = \mathbf{G}(L) = 0.$$

Thus, $L(\dot{c}(t)) = \delta$ is a constant. But this constant might be zero or negative in the case when L is not positive definite. Take an arbitrary vector field $V = V(t)$ along c. Let

$$\begin{aligned} \mathbf{C}(t) &:= \mathbf{C}_{\dot{c}(t)}(V(t), V(t), V(t)), \\ \mathbf{L}(t) &:= \mathbf{L}_{\dot{c}(t)}(V(t), V(t), V(t)), \\ \mathbf{E}(t) &:= \mathbf{E}_{\dot{c}(t)}(V(t), V(t)), \\ \mathbf{B}(t) &:= \mathbf{B}_{\dot{c}(t)}(V(t), V(t), V(t)). \end{aligned}$$

$\mathbf{C}(t)$ and $\mathbf{L}(t)$ are functions of t and $\mathbf{B}(t)$ is a vector field along c.

Our first result is as follows.

Proposition 11.2.1 *Assume that (M, L) is a Finsler space with constant curvature λ. Then for any geodesic c with $L(\dot{c}) = \delta$ and any vector field $V(t)$ along c,*

$$\mathbf{C}(t) = a\mathbf{s}_{\lambda_o}(t) + b\mathbf{c}_{\lambda_o}(t), \tag{11.32}$$

$$\mathbf{L}(t) = a\mathbf{c}_{\lambda_o}(t) - \lambda_o b\mathbf{s}_{\lambda_o}(t), \tag{11.33}$$

$$\mathbf{E}(t) = e, \tag{11.34}$$

$$\mathbf{B}(t) = 2\lambda\Big[\int \Big(a\mathbf{s}_{\lambda_o}(t) + b\mathbf{c}_{\lambda_o}(t)\Big)dt\Big]\dot{c}(t) + E(t). \tag{11.35}$$

Here a, b, e are constants, $\lambda_o := \lambda\delta$ and $E(t)$ is a parallel vector field along c perpendicular to $\dot{c}(t)$ with respect to $g_{\dot{c}(t)}$.

Proof: It follows from (11.26) that

$$L_{jml;k}y^k = -\lambda L C_{jlm}. \tag{11.36}$$

For simplicity, let

$$\mathbf{B}'(t) := \mathrm{D}_{\dot{c}}\mathbf{B}(t).$$

By (10.16), (11.28), (11.27) and (11.36), we obtain the following system

$$\mathbf{C}'(t) = \mathbf{L}(t), \tag{11.37}$$
$$\mathbf{L}'(t) = -\lambda_o \mathbf{C}(t), \tag{11.38}$$
$$\mathbf{E}'(t) = 0, \tag{11.39}$$
$$\mathbf{B}'(t) = 2\lambda \mathbf{C}(t)\dot{c}(t). \tag{11.40}$$

The proposition follows from the above equations. Q.E.D.

From Proposition 11.2.1, we immediately obtain the following

Theorem 11.2.2 ([AZ2]) Let (M, F) be a complete positive definite Finsler space of constant curvature $\lambda < 0$. If $\mathbf{C}$ does not grow exponentially, then F is Riemannian. In particular, any compact positive definite Finsler space (M, F) with negative constant curvature must be Riemannian.

Proof: Take an arbitrary geodesic c defined on $(-\infty, \infty)$ with $\delta = F(\dot{c}) > 0$ and an arbitrary parallel vector field $V(t)$ along c. By assumption $\lambda_o = \lambda\delta < 0$. Define $\mathbf{C}(t)$ as above. By Proposition 11.2.1, we obtain

$$\mathbf{C}(t) = a\mathbf{s}_{\lambda_o}(t) + b\mathbf{c}_{\lambda_o}(t).$$

Clearly $\mathbf{C}(t)$ is unbounded on $(-\infty, \infty)$ if $a \neq 0$ or $b \neq 0$. By assumption,

$$\|\mathbf{C}\| = \sup_{y\neq 0, v\in TM} \frac{|\mathbf{C}_y(v,v,v)|}{[g_y(v,v)]^{\frac{3}{2}}} < \infty.$$

This implies that

$$|\mathbf{C}(t)| \leq \|\mathbf{C}\| Q^{\frac{3}{2}} < \infty,$$

where $Q = g_{\dot{c}}(V(t), V(t))$ is a positive constant by Lemma 7.3.2, since $V(t) \neq 0$ is parallel along c. We conclude that $a = b = 0$ and $\mathbf{C}(t) = 0$. Since the geodesic $c(t)$ and the parallel vector field $V(t)$ along c are taken arbitrarily, we conclude that $\mathbf{C} = 0$, i.e., L is Riemannian. Q.E.D.

From (11.34), we see that on a Finsler space of constant curvature, the mean Berwald curvature is constant along geodesics. A natural question arises: is there any non-trivial Finsler metrics of constant curvature with vanishing mean Berwald curvature ? So far, we have only found one example. First, by Example 5.2.3, there exist a lots of Randers metrics on S^3 with $\mathbf{S} = 0$. By (6.13), such Randers metrics must satisfy $\mathbf{E} = 0$. From this family of Randers metrics with $\mathbf{S} = 0$, we have found a special family of Randers metrics with constant curvature $\mathbf{K} = 1$. See Example 11.2.1 below.

Example 11.2.1 ([BaSh2]) On the Lie group $Sp(1) = S^3$, there are three linearly independent right invariant 1-forms ζ^1, ζ^2, ζ^3 satisfying

$$d\zeta^1 = 2\zeta^2 \wedge \zeta^3, \qquad d\zeta^2 = 2\zeta^3 \wedge \zeta^1, \qquad d\zeta^3 = 2\zeta^1 \wedge \zeta^2. \tag{11.41}$$

Let

$$\zeta_2{}^1 := \zeta^3, \qquad \zeta_3{}^1 := -\zeta^2, \qquad \zeta_3{}^2 := \zeta^1$$

and $\zeta_j{}^i = -\zeta_i{}^j$. We obtain

$$d\zeta^i = \zeta^j \wedge \zeta_j{}^i. \tag{11.42}$$

Fix an arbitrary constant $k \geq 1$. Let $\{\theta^1 := \sqrt{k}\zeta^1, \theta^2 := \zeta^2, \theta^3 := \zeta^3\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ denote the frame which is dual to $\{\theta^1, \theta^2, \theta^3\}$. Put

$$\theta_2{}^1 := \sqrt{k}\,\theta^3, \qquad \theta_3{}^1 := -\sqrt{k}\,\theta^2, \qquad \theta_3{}^2 := \Big(\frac{2}{\sqrt{k}} - \sqrt{k}\Big)\,\theta^1 \tag{11.43}$$

and $\theta_j{}^i = -\theta_i{}^j$. It follows from (11.42) that

$$d\theta^i = \theta^j \wedge \theta_j{}^i. \tag{11.44}$$

Define a Riemannian metric α and a 1-form β on S^3 by

$$\alpha(y) := \sqrt{u^2+v^2+w^2}, \qquad \beta(y) := \frac{\sqrt{k-1}}{\sqrt{k}}u,$$

where $y = u\mathbf{b}_1 + v\mathbf{b}_2 + w\mathbf{b}_3$. Warning: α depends on k ! Write $\beta := b_1\theta^1 + b_2\theta^2 + b_3\theta^3$, where

$$b_1 = \frac{\sqrt{k-1}}{\sqrt{k}}, \qquad b_2 = b_3 = 0.$$

The covariant derivatives of β are defined by

$$db_i - b_j\theta_i{}^j =: b_{ij}\theta^j.$$

An easy computation gives

$$b_{11} = 0, \qquad b_{12} = 0, \qquad b_{13} = 0,$$

$$b_{21} = 0, \qquad b_{22} = 0, \qquad b_{23} = -\sqrt{k-1}$$

$$b_{31} = 0, \qquad b_{32} = \sqrt{k-1}, \qquad b_{33} = 0.$$

Now we treat α as a special Finsler metric, hence we go to the slit tangent bundle $T\mathrm{S}^3 \setminus \{0\}$. Let

$$\omega^i := \pi^*\theta^i, \qquad \omega_j{}^i := \pi^*\theta_j{}^i.$$

The same relationship between ω^i and $\omega_j{}^i$ as in (11.43) still holds. It follows from (11.44) that

$$d\omega^i = \omega^j \wedge \omega_j{}^i. \tag{11.45}$$

Note that $\{\omega_j{}^i\}$ are just the Berwald connection forms of α on $T\mathrm{S}^3$ with respect to $\{\omega^1, \omega^2, \omega^3\}$. In order to define the covariant derivatives of a tensor on

$TS^3 \setminus \{0\}$ with respect to the Berwald connection of α, we need to expand $\{\omega^1, \omega^2, \omega^3\}$ to a coframe $\{\omega^1, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6\}$ as follows

$$\omega^{3+i} := dy^i + y^j \omega_j{}^i, \qquad i = 1, 2, 3,$$

where $(y^1, y^2, y^3) = (u, v, w)$ are functions on TS^3 determined by $y = u\mathbf{b}_1 + v\mathbf{b}_2 + w\mathbf{b}_3$. An easy computation gives

$$\begin{aligned}
\omega^4 : &= du - \sqrt{k}\, w\, \omega^2 + \sqrt{k}\, v\, \omega^3 \\
\omega^5 : &= dv + \Big(\frac{2}{\sqrt{k}} - \sqrt{k}\Big)\, w\, \omega^1 - \sqrt{k}\, u\, \omega^3 \\
\omega^6 : &= dw - \Big(\frac{2}{\sqrt{k}} - \sqrt{k}\Big)\, v\, \omega^1 + \sqrt{k}\, u\, \omega^2.
\end{aligned}$$

Since α is Riemannian, we have

$$\Omega_j{}^i := d\omega_j{}^i - \omega_j{}^k \wedge \omega_k{}^i = \frac{1}{2} R_j{}^i{}_{kl} \omega^k \wedge \omega^l.$$

The Riemann curvature coefficients $R^i_k := R_j{}^i{}_{kl} y^j y^l$ must be functions of $(y^1, y^2, y^3) = (u, v, w)$ only.

$$R^1_1 = k(v^2 + w^2), \qquad R^1_2 = -kuv, \qquad R^1_3 = -kuw$$

$$R^2_1 = -kuv \qquad R^2_2 = k(u^2 + w^2) - 4(k-1)w^2, \qquad R^2_3 = -(4-3k)vw$$

$$R^3_1 = -kuw \qquad R^3_2 = -(4-3k)vw \qquad R^3_3 = k(u^2 + v^2) - 4(k-1)v^2.$$

Let

$$F := \alpha + \beta$$

F is a Randers metric on S^3. By Example 5.2.3 and (6.13), we see that F satisfies

$$\mathbf{S} = 0, \qquad \mathbf{E} = 0.$$

Thus F is weakly Berwaldian. But it is not Berwaldian, nor Landsbergian.

In what follows, we are going to compute the Riemann curvature of F. Let $\{e_i\}_{i=1}^6$ be the frame dual to $\{\omega^i\}_{i=1}^6$. Note that $\{e_1, e_2, e_3\}$ is the horizontal lift of $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\{e_4, e_5, e_6\}$ is the set of vertical tangent vectors on TS^3. The spray $\mathbf{G}$ of α is a horizontal vector field in the form

$$\mathbf{G} = ue_1 + ve_2 + we_3.$$

It follows from (5.28) that the spray $\tilde{\mathbf{G}}$ of F is in the form

$$\tilde{\mathbf{G}} = \mathbf{G} - 2\mathbf{Q},$$

where $\mathbf{Q} = Q^1 e_4 + Q^2 e_3 + Q^3 e_6$ is a vertical vector field on TS^3 given by

$$Q^i = \frac{1}{2} b_{jk} y^j y^k \frac{y^i}{F} + \frac{1}{2}\alpha\Big(b_{il} - b_{li}\Big) y^l - \frac{1}{2}\alpha\Big(b_{jl} - b_{lj}\Big) b^j y^l \frac{y^i}{F},$$

where $(y^1, y^2, y^3) = (u, v, w)$. A direct computation gives

$$Q^1 = 0, \qquad Q^2 = -\sqrt{k-1}\, w\alpha, \qquad Q^3 = \sqrt{k-1}\, v\alpha. \tag{11.46}$$

Note that Q^i are functions of (u, v, w) only.

By Lemma 8.1.4, the Riemann curvature coefficients of $\tilde{\mathbf{G}}$ are related to that of $\mathbf{G}$ by

$$\tilde{R}^i_k = R^i_k + H^i_k. \tag{11.47}$$

where

$$H^i_k := 2Q^i_{;k} - Q^i_{\cdot k;j} y^j + 2Q^j Q^i_{\cdot j \cdot k} - Q^i_{\cdot j} Q^j_{\cdot k}. \tag{11.48}$$

Here the covariant derivatives of $\mathbf{Q}$ with respect to α are defined by

$$dQ^i + Q^j \omega_j^{\ i} = Q^i_{;k}\omega^k + Q^i_{\cdot k}\omega^{n+k}.$$

$$dQ^i_{\cdot k} + Q^j_{\cdot k}\omega_j^{\ i} - Q^i_{\cdot j}\omega_k^{\ j} = Q^i_{\cdot k;l}\omega^l + Q^i_{\cdot k \cdot l}\omega^{n+l}.$$

By a direct computation, we obtain

$$Q^1_{;1} = 0, \qquad Q^1_{;2} = -\sqrt{k}\sqrt{k-1}\, v\alpha, \qquad Q^1_{;3} = -\sqrt{k}\sqrt{k-1}\, w\alpha,$$

$$Q^2_{;1} = 0, \qquad Q^2_{;2} = \sqrt{k-1}\, u\alpha, \qquad Q^2_{;3} = 0,$$

$$Q^3_{;1} = 0, \qquad Q^3_{;2} = 0, \qquad Q^3_{;3} = \sqrt{k}\sqrt{k-1}\, u\alpha.$$

$$Q^1_{\cdot 1} = 0, \qquad Q^1_{\cdot 2} = 0, \qquad Q^1_{\cdot 3} = 0,$$

$$Q^2_{\cdot 1} = -\sqrt{k-1}\, w\alpha_u, \qquad Q^2_{\cdot 2} = -\sqrt{k-1}\, w\alpha_v, \qquad Q^2_{\cdot 3} = -\sqrt{k-1}\, (\alpha + w\alpha_w),$$

$$Q^3_{\cdot 1} = \sqrt{k-1}\, v\alpha_u, \qquad Q^3_{\cdot 2} = \sqrt{k-1}\, (\alpha + v\alpha_v), \qquad Q^3_{\cdot 3} = \sqrt{k-1}\, v\alpha_w.$$

Similarly, we obtain all formulas for $Q^i_{\cdot k;l}$ and $Q^i_{;k \cdot l}$. We omit the details here. Plugging them into (11.48) yields

$$\begin{aligned}
H^1_1 &= \sqrt{k}\sqrt{k-1}\Big(v^2 + w^2\Big)u\alpha^{-1} \\
H^1_2 &= -\sqrt{k}\sqrt{k-1}\, u^2 v\alpha^{-1} \\
H^1_3 &= -\sqrt{k}\sqrt{k-1}\, u^2 w\alpha^{-1} \\
H^2_1 &= -(k-1)uv - \sqrt{k}\sqrt{k-1}\Big(2u^2 + v^2 + w^2\Big)v\alpha^{-1} \\
H^2_2 &= (k-1)\Big(u^2 + 4w^2\Big) + 2\sqrt{k}\sqrt{k-1}\, u\alpha \\
H^2_3 &= -4(k-1)vw - \sqrt{k}\sqrt{k-1}\, uvw\alpha^{-1} \\
H^3_1 &= -(k-1)uw - \sqrt{k}\sqrt{k-1}(2u^2 + v^2 + w^2)w\alpha^{-1} \\
H^3_2 &= -4(k-1)vw - \sqrt{k}\sqrt{k-1}\, uvw\alpha^{-1} \\
H^3_3 &= (k-1)\Big(u^2 + 4v^2\Big) + 2\sqrt{k}\sqrt{k-1}\, u\alpha
\end{aligned}$$

We add R^i_k and H^i_k together and find a simple formula for $\tilde{R}^i_k$,

$$\tilde{R}^i_k = kF^2\Big\{\delta^i_k - F^{-1}F_{y^j}y^i\Big\}$$

where $(y^1, y^2, y^3) = (u, v, w)$. Thus F is of constant curvature $\mathbf{K} = k \geq 1$. Normalizing F, we obtain a family of Randers metrics $F' = \frac{1}{\sqrt{k}}F$ on S^3 with constant curvature $\mathbf{K} = 1$. ♯

There are some other interesting Randers metrics of constant curvature.

Example 11.2.2 On the unit ball B^n in $\mathbb{R}^n$, the Funk metric is given by

$$F(y) := \frac{\sqrt{|y|^2 - \Big(|x|^2|y|^2 - \langle x, y\rangle^2\Big)}}{1-|x|^2} + \frac{\langle x, y\rangle}{1-|x|^2}.$$

where $|\cdot|$ and $\langle,\rangle$ denote the standard Euclidean norm and inner product in R^n. The Funk metric F has constant curvature $\mathbf{K} = -1/4$. Moreover, F is only positively complete. The Cartan torsion $\mathbf{C}$ is bounded too. More precisely, at any point $x \in \mathrm{B}^n$,

$$\|\mathbf{C}\|_x := \sup_{y,v\in T_xM} \frac{|\mathbf{C}_y(v, v, v)|}{[g_y(v, v)]^{\frac{3}{2}}} \leq \frac{3}{\sqrt{2}}.$$

Does this contradict Theorem 11.2.2? ♯

It is then a natural problem to classify all Randers metrics with constant curvature. This problem was first studied by M. Matsumoto [Ma2], then by Shibata-Shimada-Azuma-Yasuda [SSAY]. Finally, Yasuda-Shimada [YaSh] in 1977 successfully found a set of sufficient and necessary conditions on β for F with constant curvature. Surely, the computation is very complicated. Many years later, Shibata-Kitayama [ShKi] and Matsumoto [Ma6] verified their result in a different approach for Randers metrics of constant curvature. Inspired by Yasuda-Shimada's result, Bao-Shen find the above family of Randers metrics on S^3 with $\mathbf{K} = 1$.

We continue to study Finsler spaces of constant curvature. Let (M, L) be a Finsler space of constant curvature λ. (11.18) simplifies to

$$R_j{}^i{}_{kl}y^l = \lambda\Big\{g_{jl}y^l\delta^i_k - g_{jk}y^i\Big\}. \tag{11.49}$$

Plugging (11.36) and (11.49) into (10.35), (10.36), (10.40) and (10.41) yield

$$\begin{aligned} C_{ijk;l;m}y^m &= \lambda\Big\{C_{jkl}g_{im} + C_{ikl}g_{jm} + C_{ijl}g_{km} + C_{ijk}g_{lm}\Big\}y^m \\ &\quad +\lambda LC_{ijk\cdot l} + L_{ijk;l}, \end{aligned} \tag{11.50}$$

$$
\begin{aligned}
C_{ijk\cdot l;m}y^m &= -C_{ijk;l} + L_{ijk\cdot l}, && (11.51)\\
L_{ijk;l;m}y^m &= \lambda\Big\{L_{jkl}g_{is} + L_{ikl}g_{js} + L_{ijl}g_{ks}\Big\}y^s \\
&\quad -\lambda L C_{ijk;l} + \lambda L L_{ijk\cdot l}, && (11.52)\\
L_{ijk\cdot l;m}y^m &= -\lambda L C_{ijk\cdot l} - L_{ijk;l} - 2\lambda C_{ijk}g_{ls}y^s. && (11.53)
\end{aligned}
$$

Let c be a geodesic with

$$\delta := L(\dot c)$$

and $V(t), W(t)$ parallel vector fields along c. Define $\bar{\mathbf{C}}(t), \tilde{\mathbf{C}}(t), \bar{\mathbf{L}}(t)$ and $\tilde{\mathbf{L}}(t)$ as in (10.49)-(10.52). When $V(t) = \dot c(t)$ or $W(t) = \dot c(t)$, the above functions take special forms. For simplicity, we assume that

$$g_{\dot c(t)}\Big(\dot c(t), V(t)\Big) = 0 = g_{\dot c(t)}\Big(\dot c(t), W(t)\Big).$$

It follows from (11.50)-(11.53) that

$$
\begin{aligned}
\bar{\mathbf{C}}'(t) &= \lambda_o\tilde{\mathbf{C}}(t) + \bar{\mathbf{L}}(t), && (11.54)\\
\tilde{\mathbf{C}}'(t) &= -\bar{\mathbf{C}}(t) + \tilde{\mathbf{L}}(t), && (11.55)\\
\bar{\mathbf{L}}'(t) &= -\lambda_o\bar{\mathbf{C}}(t) + \lambda_o\tilde{\mathbf{L}}(t), && (11.56)\\
\tilde{\mathbf{L}}'(t) &= -\lambda_o\tilde{\mathbf{C}}(t) - \bar{\mathbf{L}}(t), && (11.57)
\end{aligned}
$$

where $\lambda_o := \lambda\delta$.

Luckily, the above system is solvable. The interesting case is when $\lambda_o = 1$. Assume that $\lambda_o = 1$. From (11.54)-(11.57), we obtain

$$
\begin{aligned}
\bar{\mathbf{C}}(t) &= \alpha\ \sin(2t) + \beta\ \cos(2t) + c, && (11.58)\\
\tilde{\mathbf{C}}(t) &= -\beta\ \sin(2t) + \alpha\ \cos(2t) + d, && (11.59)\\
\bar{\mathbf{L}}(t) &= -\beta\ \sin(2t) + \alpha\ \cos(2t) - d, && (11.60)\\
\tilde{\mathbf{L}}(t) &= -\alpha\ \sin(2t) - \beta\ \cos(2t) + c, && (11.61)
\end{aligned}
$$

where α, β, c and d are constants.

From the above identities, we see that $\bar{\mathbf{C}}(t)$, et al are periodic functions of period π (not 2π) along geodesics for any $g_{\dot c(t)}$-orthogonal parallel vector field $V(t)$.

11.3 Riemannian Spaces of Constant Curvature

One can easily show that for a Riemannian metric g on a manifold M, the flag curvature $\mathbf{K}(P, y)$ of a flag $\{P, y\} \subset T_xM$ is independent of the direction $y \in P$. Thus $\mathbf{K}(P, y) = \mathbf{K}(P)$ depends only on the tangent plane $P \subset T_xM$. In this case, we call $\mathbf{K}(P)$ the sectional curvature of the section P. By a similar argument, one can also show that if g is of scalar curvature $\lambda = \lambda(y)$, then

$\lambda = \lambda(x)$ is a function of $x \in M$ only. In this case, the sectional curvature is constant at each point, namely,

$$\mathbf{K}(P) = \lambda(x), \qquad \forall P \subset T_x M.$$

Further, in dimension $n \geq 3$, the scalar function $\lambda = constant$. Riemannian metrics of constant curvature have been completely classified. In this section, we will discuss some examples of Riemannian metrics of constant curvature. First let us take a look at the following

Example 11.3.1 (Klein Disk) Consider the Klein metric on the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$ (see Section 2.3)

$$L = \frac{|y|^2 - \Big(|x|^2|y|^2 - \langle x, y\rangle^2\Big)}{\Big(1-|x|^2\Big)^2},$$

where $\langle , \rangle$ and $|\cdot|$ denote the standard inner product and Euclidean norm in $\mathbb{R}^n$. By a direct computation, we obtain

$$\begin{aligned} g_{ij} &= \frac{1}{1-|x|^2}\Big(\delta_{ij} + \frac{x^i x^j}{1-|x|^2}\Big), \\ g^{ij} &= \big(1-|x|^2\big)\Big(\delta_{ij} - x^i x^j\Big), \end{aligned}$$

$$2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} = \frac{3\delta_{jl}x^k + \delta_{kl}x^j}{\big(1-|x|^2\big)^2} + \frac{4x^j x^k x^l}{\big(1-|x|^2\big)^3}. \tag{11.62}$$

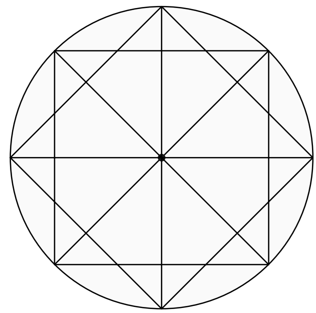
Plugging (11.62) into (8.85) gives

$$G^i = \frac{3\delta_{ij}x^k + \delta_{ik}x^j}{4\big(1-|x|^2\big)} y^j y^k = \frac{\langle x, y\rangle}{1-|x|^2} y^i. \tag{11.63}$$

We obtain the spray of L.

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2\frac{\langle x, y\rangle}{1-|x|^2}\, y^i \frac{\partial}{\partial y^i}.$$

Clearly, the geodesics of L are straight lines in $\mathbb{B}^n$.



Plugging $P = \frac{\langle x,y\rangle}{1-|x|^2}$ into (9.43), we can obtain

$$R^i_k = -\Big[L\delta^i_k - \frac{1}{2}L_{y^k}y^i\Big]. \tag{11.64}$$

Namely, L has constant curvature $\mathbf{K}(P) = -1$. ♯

Example 11.3.2 Consider the following Riemannian metric on R^n

$$L := \frac{|y|^2 + \Big(|x|^2|y|^2 - \langle x,y\rangle^2\Big)}{\Big(1+|x|^2\Big)^2}.$$

A direct computation yields

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} + 2\frac{\langle x,y\rangle}{1+|x|^2}\, y^i\frac{\partial}{\partial x^i}. \tag{11.65}$$

Plugging $P = -\frac{\langle x,y\rangle}{1+|x|^2}$ into (9.43), we obtain

$$R^i_k = L\delta^i_k - \frac{1}{2}L_{y^k}y^i. \tag{11.66}$$

Namely, L has constant curvature $\mathbf{K}(P) = 1$. ♯

One of the most important theorems in Riemannian geometry is the classification of simply connected Riemannian spaces with constant sectional curvature. This is due to E. Cartan.

Theorem 11.3.1 *For any $\lambda \in \{-1,0,1\}$ and $n \geq 2$, there is an unique complete simply connected n-dimensional Riemannian space with constant sectional curvature λ.*

Complete simply connected Riemannian spaces with constant sectional curvature are called the *(Riemannian) space forms.* We will describe them below in details.

The space form with $\lambda = 1$ is the standard unit sphere $\mathbb{S}^n = (\mathrm{S}^n, g_{\mathrm{S^n}})$ in $\mathbb{R}^{n+1}$. Let $\{p, q\}$ denote the north pole and the south pole of S^n respectively. $\mathrm{S}^n \setminus \{p, q\} \approx (0, \pi) \times \mathrm{S}^{n-1}$. $g_{\mathrm{S^n}}$ on $\mathrm{S}^n \setminus \{p, q\}$ is expressed by

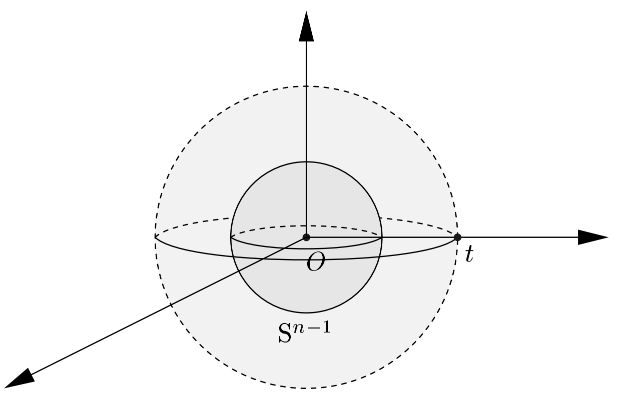
$$g_{\mathrm{S^n}} = dt \otimes dt + \sin^2 t \; g_{\mathrm{S^{n-1}}}. \tag{11.67}$$

The space form with $\lambda = 0$ is just the Euclidean space $\mathbb{R}^n = (\mathrm{R}^n, g_{\mathrm{R^n}})$ Topologically $\mathrm{R}^n \setminus \{0\} \approx (0, \infty) \times \mathrm{S}^{n-1}$. $g_{\mathrm{R^n}}$ on $\mathrm{R}^n \setminus \{0\}$ is expressed by

$$g_{\mathrm{R^n}} = dt \otimes dt + t^2 g_{\mathrm{S^{n-1}}}. \tag{11.68}$$

The space form with $\lambda = -1$ is called the *hyperbolic space* $\mathbb{H}^n = (\mathrm{R}^n, g_{\mathrm{H^n}})$. $g_{\mathrm{H^n}}$ on $\mathrm{R}^n \setminus \{0\} \approx (0, \infty) \times \mathrm{S}^{n-1}$ is expressed by

$$g_{\mathrm{H^n}} = dt \otimes dt + \sinh^2(t) g_{\mathrm{S^{n-1}}}. \tag{11.69}$$



Example 11.3.3 Consider the following Riemannian metric on a surface

$$L := a(x, y)\, u^2 + c(x, y)\, v^2. \tag{11.70}$$

An easy computation yields

$$\mathbf{K} = -\frac{1}{2\sqrt{ac}} \Big[\Big(\frac{c_x}{\sqrt{ac}} \Big)_x + \Big(\frac{a_y}{\sqrt{ac}} \Big)_y \Big]. \tag{11.71}$$

Take a look at a special surface $\varphi : (a, b) \times \mathrm{S}^1 \to \mathrm{R}^3$ defined by

$$\varphi(x, y) := \Big(f(x) \cos y, \; f(x) \sin y, \; x \Big).$$

The induced Riemannian metric is given by

$$L = \Big(1 + f'(x)^2\Big)u^2 + f(x)^2 v^2$$

It follows from (11.71) that

$$\mathbf{K} = -\frac{f''(x)}{f(x)\Big[1 + f'(x)^2\Big]^2}.$$

(a) Take

$$f(x) = \sqrt{1 - x^2}.$$

Then $\mathbf{K} = 1$. This is the standard unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$.

(b) Take

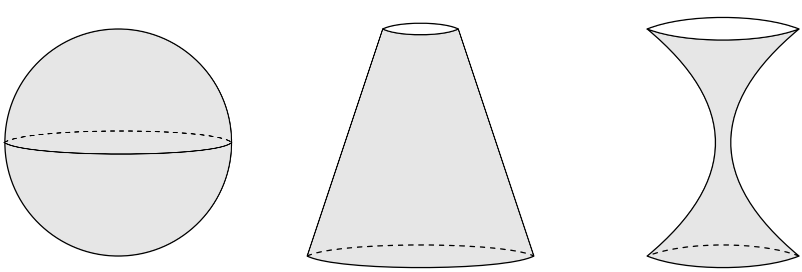
$$f(x) = ax + b.$$

Then $\mathbf{K} = 0$. The surface is a cone in $\mathbb{R}^3$.

(c) Take $f(x)$ such that

$$f''(x) = f(x)\Big[1 + f'(x)^2\Big]^2.$$

Then $\mathbf{K} = -1$.



$\mathbf{K} = 1$ $\mathbf{K} = 0$ $\mathbf{K} = -1$

♯

There are many *incomplete* Riemannian metrics of constant curvature.

Example 11.3.4 Take family of Riemannian metrics

$$g_c := dt \otimes dt + c^2 \sin^2(t) g_o,$$

on $(0, \pi) \times \mathrm{S}^1$, where $g_o = ds \otimes ds$ denotes the canonical metric on S^1. One can verify that g_c has constant curvature $\lambda = 1$ for any $c > 0$. ♯

Chapter 12

Projective Geometry

Two sprays $\mathbf{G}$ and $\tilde{\mathbf{G}}$ on a manifold are said to be *pointwise projectively related* if they have the same geodesics as point sets. For any geodesic $c(t)$ of $\mathbf{G}$, there is an orientation-preserving reparameterization $t = t(s)$ such that $c(s) := c(t(s))$ is a geodesic of $\tilde{\mathbf{G}}$, and vice versa. In this chapter, we will show that two sprays $\mathbf{G}$ and $\tilde{\mathbf{G}}$ on a manifold are pointwise projectively related if and only if there is a scalar function P on $TM \setminus \{0\}$ such that

$$\tilde{\mathbf{G}} = \mathbf{G} - 2P\,\mathbf{Y}. \tag{12.1}$$

Then we prove the Rapcsák theorem on projectively related Finsler metrics. This remarkable theorem plays an important role in the projective geometry of Finsler spaces. See [Th3] for a systematic survey on the early development in this field.

12.1 Projectively Related Sprays

Let $\mathbf{G}(y) = y^i \frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}$ be a spray on a manifold M. Geodesics are locally characterized by the following system

$$\frac{d^2 c^i}{dt^2} + 2G^i\Big(\frac{dc}{dt}\Big) = 0, \tag{12.2}$$

where $(c^i(t))$ denote the coordinates of $c(t)$. Consider another spray $\tilde{\mathbf{G}} = y^i \frac{\partial}{\partial x^i} - 2\tilde{G}^i(y)\frac{\partial}{\partial y^i}$ on M, where

$$\tilde{G}^i(y) = G^i(y) + P(y)\,y^i. \tag{12.3}$$

The scalar function P must satisfy the following homogeneity condition

$$P(\lambda y) = \lambda P(y), \qquad \lambda > 0. \tag{12.4}$$

Assume that $c(t)$ is a geodesic of $\mathbf{G}$, hence the coordinates $(c^i(t))$ of $c(t)$ satisfy (12.2). Take a new parameter s determined by

$$\frac{d^2 s}{dt^2} = 2P\Big(\frac{dc}{dt}\Big)\frac{ds}{dt}, \qquad \frac{ds}{dt} > 0.$$

Then $c(s) := c(t(s))$ satisfies

$$\begin{aligned}\frac{d^2c^i}{ds^2} &= \frac{-2G^i\Big(\frac{dc}{dt}\Big)\frac{ds}{dt} - \frac{dc^i}{dt}\frac{d^2s}{dt^2}}{\Big(\frac{ds}{dt}\Big)^3} \\ &= \frac{-2G^i\Big(\frac{dc}{dt}\Big)\frac{ds}{dt} - 2\frac{dc^i}{dt}P\Big(\frac{dc}{dt}\Big)\frac{ds}{dt}}{\Big(\frac{ds}{dt}\Big)^3} \\ &= -2\tilde{G}^i\Big(\frac{dc}{ds}\Big).\end{aligned}$$

Thus $c(s)$ is a geodesic of $\tilde{\mathbf{G}}$. This implies that $\tilde{\mathbf{G}}$ and $\mathbf{G}$ are pointwise projectively related.

The converse is true too. Assume that sprays $\tilde{\mathbf{G}}$ and $\mathbf{G}$ have the same geodesics as point sets. Take an arbitrary geodesic c in M. Let t and s denote the geodesic parameters of c with respect to $\mathbf{G}$ and $\tilde{\mathbf{G}}$ respectively. Parameterize c by a common parameter u. By (12.2) we obtain

$$\frac{d^2c^i}{du^2} + 2G^i(\frac{dc}{du}) = \phi(u)\frac{dc^i}{du}, \tag{12.5}$$

$$\frac{d^2c^i}{du^2} + 2\tilde{G}^i(\frac{dc}{du}) = \tilde{\phi}(u)\frac{dc^i}{du}, \tag{12.6}$$

where

$$\phi(u) := \frac{\frac{d^2u}{dt^2}}{\Big(\frac{du}{dt}\Big)^2}, \qquad \tilde{\phi}(u) := \frac{\frac{d^2u}{ds^2}}{\Big(\frac{du}{ds}\Big)^2}.$$

The difference of (12.5) and (12.6) is

$$2\tilde{G}^i(\frac{dc}{du}) - 2G^i(\frac{dc}{du}) = (\tilde{\phi} - \phi)\frac{dc^i}{du}.$$

Since the above equations hold for any geodesic, we conclude that (12.3) holds for some function P on TM satisfying (12.4). We have proved the following

Lemma 12.1.1 *Let $(M, \mathbf{G})$ be a spray space. A spray $\tilde{\mathbf{G}}$ is pointwise projective to $\mathbf{G}$ if and only if there is a scalar function P on $TM \setminus \{0\}$ such that*

$$\tilde{\mathbf{G}} = \mathbf{G} - 2P\mathbf{Y}.$$

Consider a spray $\mathbf{G}$ on a manifold M. Fix a standard local coordinate system (x^i, y^i) in TM and write $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}$. Let

$$\Phi^a(s,\eta,\xi) := 2\xi^a G^1(s,\eta,1,\xi) - 2G^a(s,\eta,1,\xi), \qquad a = 2,\cdots,n, \tag{12.7}$$

where $\eta = (\eta^2,\cdots,\eta^n)$ and $\xi = (\xi^2,\cdots,\xi^n)$. Let

$$\hat{G}^1 := 0, \qquad \hat{G}^a := -\frac{1}{2}y^1y^1\Phi^a\Big(x^1,\cdots,x^n,\frac{y^2}{y^1},\cdots,\frac{y^n}{y^1}\Big).$$

It is easy to verify that the local spray $\hat{\mathbf{G}} := y^i\frac{\partial}{\partial x^i} - 2\hat{G}^i(y)\frac{\partial}{\partial y^i}$ is pointwise projective to $\mathbf{G}$.

Construct another local spray as follows

$$\Pi := y^i \frac{\partial}{\partial x^i} - 2\Pi^i(y)\frac{\partial}{\partial y^i}, \tag{12.8}$$

where

$$\Pi^i(y) := G^i(y) - \frac{1}{n+1}N^m_m(y)\, y^i. \tag{12.9}$$

Clearly, Π is also pointwise projective to $\mathbf{G}$. Π is called the *local projective spray* associated with $\mathbf{G}$. Π has the following important property: If a spray $\tilde{\mathbf{G}}$ is pointwise projective to $\mathbf{G}$, then the local projective spray $\tilde{\Pi}$ associated with $\tilde{\mathbf{G}}$ is equal to Π in the same standard local coordinate system ! More important, any (local) invariant expressed in terms of Π^i is a (local) projective invariant. However, "projective invariants" defined by Π might not be globally defined. To avoid this problem, we need an additional structure on the manifold, that is, a volume form.

Fix a volume form $d\mu = \sigma(x)dx^1\cdots dx^n$ on M. For a spray $\mathbf{G}$ on $(M, d\mu)$, let $\mathbf{S}$ denote the S-curvature of $(\mathbf{G}, d\mu)$. Define

$$\tilde{\mathbf{G}} := \mathbf{G} + \frac{2\mathbf{S}}{n+1}\mathbf{Y}. \tag{12.10}$$

$\tilde{\mathbf{G}}$ is a globally defined spray on M ! We call $\tilde{\mathbf{G}}$ the *projective spray* associated with $\mathbf{G}$ on $(M, d\mu)$. In a standard local coordinate system (x^i, y^i) in TM, $\tilde{\mathbf{G}} = y^i\frac{\partial}{\partial x^i} - 2\tilde{G}^i(y)\frac{\partial}{\partial y^i}$ is given by

$$\tilde{G}^i(y) := G^i(y) - \frac{\mathbf{S}(y)}{n+1}y^i. \tag{12.11}$$

By the definition of the S-curvature, we have

$$\tilde{G}^i(y) = \Pi^i(y) + \frac{1}{n+1}\frac{y^m}{\sigma}\frac{\partial\sigma}{\partial x^m}\, y^i, \tag{12.12}$$

where Π^i are given in (12.9). Thus $\tilde{\mathbf{G}}$ depends only on the pointwise projective class of $\mathbf{G}$ for a fixed volume form $d\mu$. This implies that any invariant defined by $\tilde{\mathbf{G}}$ is a globally defined projective invariant. However, the invariants defined by $\tilde{\mathbf{G}}$ possibly depend on the volume form $d\mu$ too. As long as the invariant is independent of the volume form $d\mu$, it is a globally defined projective invariant of $\mathbf{G}$.

Example 12.1.1 Let ϕ and ψ be C^∞ functions on an open subset $\Omega \subset \mathrm{R}^2$ satisfying $\phi^2+\psi^2<1$. Let (x,y,u,v) denote the standard global coordinate system in $T\Omega = \Omega \times R^2$. Let

$$F := \sqrt{u^2+v^2} + \phi(x,y)\, u + \psi(x,y)\, v. \tag{12.13}$$

By (5.28), F is pointwise projectively related to the following spray

$$\mathbf{G} := u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - \sqrt{u^2+v^2}(\phi_y - \psi_x)\Big(v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}\Big). \tag{12.14}$$

♯

The Riemann curvature of sprays is determined by geodesics. If two sprays are pointwise projectively related, then the Riemann curvatures are related by a simple equation.

Lemma 12.1.2 *Let* $\mathbf{G}$ *and* $\tilde{\mathbf{G}} := \mathbf{G} - 2P\mathbf{Y}$ *be sprays on an* n*-manifold* M*. The Riemann curvatures are related by*

$$\tilde{\mathbf{R}}_y = \mathbf{R}_y + \Xi(y)\, I + \tau_y\, y, \tag{12.15}$$

where $\tau_y \in T_x^*M$ *with* $\tau_y(y) = -\Xi(y)$*. Hence, the Ricci scalars* $\tilde{R} = \frac{1}{n-1}\widetilde{\mathbf{Ric}}$ *and* $R = \frac{1}{n-1}\mathbf{Ric}$ *are related by*

$$\tilde{R}(y) = R(y) + \Xi(y). \tag{12.16}$$

Proof: By (8.14), we obtain the following identity in a local coordinate system

$$\tilde{R}^i_k = R^i_k + \Xi\, \delta^i_k + \tau_k\, y^i \tag{12.17}$$

where

$$\Xi := P^2 - P_{;k}y^k \tag{12.18}$$

$$\tau_k := 3(P_{;k} - PP_{\cdot k}) + \Xi_{\cdot k}. \tag{12.19}$$

Observe that

$$\tau_k y^k = 3(P_{;k}y^k - P^2) + 2\Xi = -\Xi.$$

It follows from (12.17) that

$$\tilde{R} = R + \frac{1}{n-1}\big(n\Xi - \Xi\big) = R + \Xi. \tag{12.20}$$

This completes the proof. Q.E.D.

The equation (12.15) was established in [Ma3] for Finsler metrics.

Recall that a spray $\mathbf{G}$ is isotropic if the Riemann curvature is in the following form

$$\mathbf{R}_y = R(y)I + \zeta_y\, y.$$

From (12.15), we see that if a spray is pointwise projective to an isotropic spray, then it must be isotropic too.

By (12.15)-(12.19), we immediately obtain the following

Proposition 12.1.3 *Let $(M, \mathbf{G})$ be a spray space and $\tilde{\mathbf{G}} := \mathbf{G} - 2P\mathbf{Y}$ a projectively related spray.*

(a) $\tilde{\mathbf{R}} = \mathbf{R}$ if and only if P satisfies

$$P_{;k} - PP_{\cdot k} = 0. \tag{12.21}$$

(b) $\widetilde{\mathbf{Ric}} = \mathbf{Ric}$ if and only if P satisfies

$$y^k P_{;k} = P^2. \tag{12.22}$$

The special case of Proposition 12.1.3 has been proved in Example 9.3.1. Comparing (12.21) with (2.41), we make the following

Definition 12.1.4 Let $(M, \mathbf{G})$ be a spray space. Assume that a function P on TM is C^∞ on $TM \setminus \{0\}$ satisfying

$$P(\lambda y) = \lambda P(y), \qquad \forall \lambda > 0.$$

(a) P is called a *Funk function* if it satisfies the following system of PDEs

$$P_{;k} = PP_{\cdot k}.$$

(b) P is called a *weak Funk function* if it satisfies the following PDE

$$y^k P_{;k} = P^2.$$

A natural question is whether or not there always exist non-trivial Funk functions on a spray space. The following proposition tells us that there are no non-trivial Funk functions on compact spray spaces.

Proposition 12.1.5 *Let P be a weak Funk function on a spray space $(M, \mathbf{G})$. Then for any geodesic $c(t)$ in $(M, \mathbf{G})$ with $\dot{c}(0) = y$,*

$$P(\dot{c}) = \frac{P(y)}{1 - P(y)t}. \tag{12.23}$$

In addition, if $\mathbf{G}$ is complete, then $P = 0$.[1]

[1] There are imcomplete sprays on a compact manifold. In the published book, we assume that M is compact instead of $\mathbf{G}$ is complete. Actually, we just need the completeness of $\mathbf{G}$.

Proof: Let $P(t) := P(\dot{c}(t))$. We have

$$P'(t) = P_{;k}(\dot{c}(t))\dot{c}^k(t),$$

where $(c^k(t))$ denote the coordinates of $c(t)$. Hence

$$P^2(t) - P'(t) = 0.$$

Solving the above ODE, we immediately obtain (12.23). Q.E.D.

For sprays $\mathbf{G}$ and $\tilde{\mathbf{G}} = \mathbf{G} - 2P\mathbf{Y}$, $\widetilde{\mathbf{Ric}} = \mathbf{Ric}$ if and only if P is a weak Funk function (Proposition 12.1.3). Proposition 12.1.5 can be generalized to the following

Proposition 12.1.6 *Let* $\mathbf{G}$ *be a positively complete spray space. Assume that a spray* $\tilde{\mathbf{G}} = \mathbf{G} - 2P\mathbf{Y}$ *satisfies*

$$\mathbf{Ric} \geq \widetilde{\mathbf{Ric}},$$

then the projective factor P *satisfies*

$$P \leq 0.$$

In addition, if $\mathbf{G}$ *and* $\tilde{\mathbf{G}}$ *are both reversible or* $\mathbf{G}$ *is complete, then* $\tilde{\mathbf{G}} = \mathbf{G}$.

Proof: Assume that $P(v) > 0$. Let c be a geodesic in $(M, \mathbf{G})$ with $\dot{c}(0) = v$. It follows from (12.16) that $P(t) := P(\dot{c}(t))$ satisfies

$$P'(t) - P^2(t) = -\Xi(\dot{c}(t)) = R(\dot{c}(t)) - \tilde{R}(\dot{c}(t)) \geq 0. \tag{12.24}$$

Let

$$P_0(t) := \frac{P(v)}{1 - P(v)t}.$$

$P_0(t)$ satisfies

$$P_0'(t) - P_0(t)^2 = 0. \tag{12.25}$$

To compare $P(t)$ with $P_0(t)$, we define

$$h(t) := e^{-\int_0^t \left[P(s)+P_0(s)\right] ds} \Big\{ P(t) - P_0(t) \Big\}.$$

It follows from (12.24) and (12.25) that

$$h'(t) = e^{-\int_0^t \left[P(s)+P_0(s)\right] ds} \Big\{ P'(t) - P(t)^2 - P_0'(t) + P_0(t)^2 \Big\} \geq 0.$$

Since $h(0) = 0$, we conclude that

$$P(t) - P_0(t) \geq 0, \qquad t \geq 0.$$

Let $t_o := 1/P(v) > 0$. Then

$$P(\dot{c}(t_o)) = \lim_{t\to t_o^-} P(t) \geq \lim_{t\to t_o^-} P_0(t) = +\infty.$$

This contradicts our assumption. Thus $P(v) \leq 0$.

If $\mathbf{G}$ is also negatively complete, then $P(v) \geq 0$ by a similar argument. Hence $P(v) = 0$.

If both $\mathbf{G}$ and $\tilde{\mathbf{G}}$ are reversible, then P satisfies

$$P(-v) = -P(v) \leq 0.$$

Thus $P(v) = 0$. Q.E.D.

Let $\mathbf{G}_0 := y^i \frac{\partial}{\partial x^i}$ denote the canonical flat spray on R^n and $\mathbf{G}$ a reversible spray on R^n with $\mathbf{Ric} \geq 0$. By Proposition 12.1.6, we conclude that if $\mathbf{G}$ is complete, then it can not be pointwise projective to $\mathbf{G}_0$ unless $\mathbf{G} = \mathbf{G}_0$. However, there are many *incomplete* sprays on R^n with $\mathbf{Ric} \geq 0$, which are pointwise projective to $\mathbf{G}_0$.

Example 12.1.2 Consider the following spray on R^n

$$\mathbf{G} = \mathbf{G}_0 + 2\frac{\langle x, y\rangle}{1+|x|^2}\mathbf{Y}. \tag{12.26}$$

A simple computation yields that $\mathbf{Ric} > 0$. Note that $\mathbf{G}$ is incomplete and $\mathbf{G}_0 \neq \mathbf{G}$. ♯

Let $\mathbf{G}$ be a reversible spray on R^n with $\mathbf{Ric} \leq 0$. By Proposition 12.1.6, we conclude that $\mathbf{G}$ can not be pointwise projective to $\mathbf{G}_0$ unless $\mathbf{G} = \mathbf{G}_0$. However, there are many reversible sprays on a proper domain $\Omega \subset \mathrm{R}^n$ with $\mathbf{Ric} \leq 0$, which are pointwise projective to $\mathbf{G}_0$.

Example 12.1.3 Consider the following spray on the unit ball $\mathrm{B}^n \subset \mathrm{R}^n$

$$\mathbf{G} = \mathbf{G}_0 - 2\frac{\langle x, y\rangle}{1-|x|^2}\,\mathbf{Y}. \tag{12.27}$$

An easy computation yields

$$\mathbf{Ric} < 0.$$

Note that B^n is a proper domain of R^n and $\mathbf{G} \neq \mathbf{G}_0$. ♯

12.2 Projectively Related Finsler Metrics

A Finsler metric on an open subset in R^n is called a *projective Finsler metric* if it is pointwise projective to the Euclidean metric. The problem of characterizing and studying projective Finsler metrics is known as *Hilbert's Fourth Problem.* More general, given a Finsler metric on a manifold, we would like to determine all Finsler metrics on the manifold whose geodesics coincide with the geodesics of the given one as set points.

A. Rapcsák [Rap] proved the following important results.

Lemma 12.2.1 *Let (M, L) be a Finsler space and $\tilde{L}$ a Finsler metric on M. Then the spray coefficients $\tilde{G}^i$ and G^i satisfy*

$$\tilde{G}^i = G^i + \frac{1}{4}\tilde{g}^{il}\Big\{\tilde{L}_{;k\cdot l}y^k - \tilde{L}_{;l}\Big\}, \tag{12.28}$$

where $\tilde{L}_{;l}$ and $\tilde{L}_{;k\cdot l}$ are the covariant derivatives of $\tilde{L}$ with respect to L given by

$$\tilde{L}_{;k} := \tilde{L}_{x^k} - N^l_k \tilde{L}_{y^l}, \qquad \tilde{L}_{;k\cdot l} := (\tilde{L}_{;k})_{y^l}.$$

Proof: Observe that

$$\tilde{L}_{;k\cdot l}y^k - \tilde{L}_{;l} = \tilde{L}_{x^k y^l}y^k - \tilde{L}_{x^l} - 4\tilde{g}_{li}G^i.$$

Then (12.28) follows immediately. Q.E.D.

Theorem 12.2.2 (Rapcsák) *Let (M, L) be a Finsler space. A Finsler metric $\tilde{L}$ on M is pointwise projective to L if and only if*

$$\tilde{L}_{;k\cdot l}y^k - \tilde{L}_{;l} = \frac{\tilde{L}_{;k}y^k}{2\tilde{L}}\tilde{L}_{\cdot l}. \tag{12.29}$$

In this case, the spray coefficients are related by $\tilde{G}^i = G^i + Py^i$, where

$$P = \frac{\tilde{L}_{;k}y^k}{4\tilde{L}}. \tag{12.30}$$

Proof: Suppose that $\tilde{L}$ is pointwise projective to L. By Lemma 12.1.1, the induced sprays $\tilde{\mathbf{G}} = y^i\frac{\partial}{\partial x^i} - 2\tilde{G}^i(y)\frac{\partial}{\partial y^i}$ and $\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}$ satisfy

$$\tilde{G}^i(y) = G^i(y) + P(y)y^i. \tag{12.31}$$

where P is a positively homogeneous function on $TM \setminus \{0\}$. By (12.28), we obtain

$$\tilde{L}_{;k\cdot l}y^k - \tilde{L}_{;l} = 4P\ \tilde{g}_{il}y^i = 2P\ \tilde{L}_{\cdot l}. \tag{12.32}$$

It follows from the homogeneity of $\tilde{L}_{;k}$ that

$$\tilde{L}_{;k\cdot l}y^l = 2\tilde{L}_{;k}.$$

This implies

$$(\tilde{L}_{;k\cdot l}y^k - \tilde{L}_{;l})y^l = 2\tilde{L}_{;k}y^k - \tilde{L}_{;l}y^l = \tilde{L}_{;k}y^k.$$

Contracting (12.32) with y^l yields

$$(\tilde{L}_{;k\cdot l}y^k - \tilde{L}_{;l})y^l = 2P\,\tilde{L}_{\cdot l}y^l = 4P\,\tilde{L}.$$

This gives (12.30). Plugging (12.30) into (12.32) gives (12.29). The converse is trivial, so is omitted. Q.E.D.

Corollary 12.2.3 *Let $\tilde{L} = \tilde{L}(x,y)$ be a Finsler metric on an open subset $\mathcal{U} \subset$ R^n. Then $\tilde{L}$ is pointwise projective to the standard Euclidean metric if and only if*

$$\tilde{L}_{x^k y^l}y^k - \tilde{L}_{x^l} = \frac{\tilde{L}_{x^k}y^k}{2\tilde{L}}\tilde{L}_{y^l}. \tag{12.33}$$

In this case, the spray coefficients $\tilde{G}^i$ of $\tilde{L}$ are given by $\tilde{G}^i = \frac{\tilde{L}_{x^k}y^k}{4\tilde{L}}y^i$.

Let L and $\tilde{L}$ be Finsler metrics on an n-dimensional manifold M. Assume that $\tilde{L}$ is pointwise projective to L. Then $\tilde{L}$ and L satisfy (12.29). The spray coefficients of L and $\tilde{L}$ are related by $\tilde{G}^i = G^i + Py^i$, where P is given by (12.30). Recall (12.15) and (12.16)

$$\tilde{\mathbf{R}}_y = \mathbf{R}_y + \Xi(y)I + \tau_y\, y, \tag{12.34}$$

$$\tilde{R}(y) = R(y) + \Xi(y), \tag{12.35}$$

where

$$\Xi(y) := P^2 - P_{;l}y^l, \tag{12.36}$$

$$\tau_y(u) := 3\big(P_{;k} - PP_{\cdot k}\big)u^k + \Xi_{\cdot k}u^k. \tag{12.37}$$

Let $\lambda(y) := R(y)/L(y)$ and $\tilde{\lambda}(y) := \tilde{R}(y)/\tilde{L}(y)$. (12.35) is equivalent to the following equation

$$\tilde{\lambda}(y)\tilde{L}(y) - \lambda(y)L(y) = \Xi(y) = \Big[\frac{\tilde{L}_{;k}y^k}{4\tilde{L}}\Big]^2 - \Big[\frac{\tilde{L}_{;k}y^k}{4\tilde{L}}\Big]_{;l}y^l. \tag{12.38}$$

Theorem 12.2.4 ([Sz2][MaWe]) *Let (M,L) be an n-dimensional Finsler space and $\tilde{L}$ another Finsler metric pointwise projective to L. Suppose that L is of scalar curvature $\lambda(y)$. Then $\tilde{L}$ is of scalar curvature $\tilde{\lambda}(y)$ which is given by (12.38).*

Proof: By assumption, L is of scalar curvature $\lambda(y)$, namely, the Riemann curvature is in the form

$$\mathbf{R}_y = \lambda(y)L(y)I + \zeta_y\, y. \tag{12.39}$$

Plugging (12.39) into (12.34) yields

$$\tilde{\mathbf{R}}_y = \tilde{\lambda}(y)\tilde{L}(y)I + \tilde{\zeta}_y\, y, \tag{12.40}$$

where $\tilde{\lambda}(y)\tilde{L}(y) := \lambda(y)L(y) + \Xi(y)$ and $\tilde{\zeta}_y := \zeta_y + \tau_y$. Q.E.D.

For a Riemannian metric g on a manifold of dimension $n \geq 3$, it is of scalar curvature if and only if it is of constant curvature. If g and $\tilde{g}$ both are pointwise projectively related Riemannian metrics, then in dimension $n \geq 3$, g is of constant curvature if and only if $\tilde{g}$ is of constant curvature. The same statement is also true for Einstein metrics. More precisely, we have

Theorem 12.2.5 (J. Mikes [Mik1][Mik2]) *Let (M, g) be an n-dimensional Riemannian space and $\tilde{g}$ another Riemannian metric pointwise projective to g. Suppose that g is Einsteinian, then $\tilde{g}$ must be Einsteinian.*

Now we consider positive definite Finsler metrics. As usual, we denote a positive definite Finsler metric L by its square root, $F = \sqrt{L}$.

Let (M, F) be a positive definite Finsler space and $\tilde{F}$ another positive definite Finsler metric on M. Plugging $\tilde{L} = \tilde{F}^2$ in (12.28) yields

$$\tilde{G}^i = G^i + Py^i + Q^i, \tag{12.41}$$

where

$$P = \frac{\tilde{F}_{;k}y^k}{2\tilde{F}}, \tag{12.42}$$

$$Q^i = \frac{\tilde{F}}{2}\tilde{g}^{il}\Big\{\tilde{F}_{;k\cdot l}y^k - \tilde{F}_{;l}\Big\}. \tag{12.43}$$

Here the covariant derivatives of $\tilde{F}$ are taken with respect to F. We immediately obtain the following

Theorem 12.2.6 (Rapcsák) *For two positive definite Finsler metrics F and $\tilde{F}$ on a manifold M, $\tilde{F}$ is pointwise projectively related to F if and only if $\tilde{F}$ satisfies*

$$\tilde{F}_{;k\cdot l}y^k - \tilde{F}_{;l} = 0. \tag{12.44}$$

In this case, the spray coefficients are related by $\tilde{G}^i = G^i + Py^i$, where

$$P = \frac{\tilde{F}_{;k}y^k}{2\tilde{F}}.$$

Example 12.2.1 Let $\beta = b_i(x)y^i$ be a 1-form and $F = F(y)$ a positive definite Finsler metric on a manifold M. Consider

$$\tilde{F} := F + \beta.$$

Adding a 1-form to a Finsler metric is called a *Randers change.* Suppose that $\|\beta\| := \sup_{F(y)=1} |\beta(y)| < 1$. $\tilde{F}$ is still a positive definite Finsler metric. Observe that

$$\beta_{;l} = \Big(\frac{\partial b_i}{\partial x^l} - \Gamma^s_{il} b_s\Big) y^i,$$
$$\beta_{;k\cdot l} y^k = \Big(\frac{\partial b_l}{\partial x^k} - \Gamma^s_{lk} b_s\Big) y^k.$$

Thus

$$\beta_{;k\cdot l} y^k - \beta_{;l} = \Big(\frac{\partial b_l}{\partial x^k} - \frac{\partial b_k}{\partial x^l}\Big) y^k.$$

By Lemma 4.2.2,

$$F_{;k} = F_{x^k} - N^l_k F_{y^l} = 0.$$

We obtain

$$\tilde{F}_{;k\cdot l} y^k - \tilde{F}_{;l} = \beta_{;k\cdot l} y^k - \beta_{;l} = \Big(\frac{\partial b_l}{\partial x^k} - \frac{\partial b_k}{\partial x^l}\Big) y^k.$$

Thus (12.44) holds for $\tilde{F} = F + \beta$ if and only if β is close. That is, $\tilde{F} = F + \beta$ is pointwise projective to F if and only if β is close. This result is obtained by Hashiquchi-Ichijyō [HaIc3] (see also Example 3.3.1.1 in [AnInMa]). ♯

Now we derive several conditions equivalent to (12.44). Let $\tilde{F}$ be a positive definite Finsler metric on a positive definite Finsler space (M, F). Let

$$P := \frac{\tilde{F}_{;k} y^k}{2\tilde{F}}.$$

Observe that

$$[P\tilde{F}]_{\cdot l} = \frac{1}{2}\tilde{F}_{;k\cdot l} y^k + \frac{1}{2}\tilde{F}_{;l}.$$

Thus $\tilde{F}$ satisfies (12.44) if and only if

$$\tilde{F}_{;l} = [P\tilde{F}]_{\cdot l}. \tag{12.45}$$

Suppose that $\tilde{F}$ satisfies (12.45) for some P satisfying $P(\lambda y) = \lambda P(y)$, $\forall \lambda > 0$. By contracting (12.45) with y^l, we obtain $P = \frac{\tilde{F}_{;l} y^l}{2\tilde{F}}$. This proves the following

Proposition 12.2.7 *Let (M, F) be a positive definite Finsler space and $\tilde{F}$ another positive definite Finsler metric on M. $\tilde{F}$ is pointwise projective to F if and only if there is a positively homogeneous function P on TM, i.e., $P(\lambda y) = \lambda P(y)$, $\forall \lambda > 0$, such that $\tilde{F}$ satisfies*

$$\tilde{F}_{;l} = [P\tilde{F}]_{\cdot l} \tag{12.46}$$

In this case,

$$P = \frac{\tilde{F}_{;k} y^k}{2\tilde{F}}.$$

and the spray coefficients are related by $\tilde{G}^i = G^i + P y^i$.

By Propositions 12.1.3 and 12.2.7, we immediately obtain the following

Proposition 12.2.8 *Let (M, F) be a positive definite space and $\tilde{F}$ a positive definite Finsler metric on M. Assume that $\tilde{F}$ satisfies*

$$\tilde{F}_{;l} = [P\tilde{F}]_{\cdot l}. \tag{12.47}$$

for some positively homogeneous function P on TM.

(a) $\tilde{\mathbf{R}} = \mathbf{R}$ if and only if P is a Funk function on (M, F);

(b) $\widetilde{\mathbf{Ric}} = \mathbf{Ric}$ if and only if P is a weak Funk function on (M, F).

A natural question arises: Given a (weak) Funk function P on a positive definite Finsler space (M, F), whether or not there exists a positive definite Finsler metric $\tilde{F}$ on M satisfying (12.47)? This inverse problem has not been studied yet. Let us take a look at the special case when P is a Funk function on an open domain $\Omega \subset \mathbb{R}^n$. By Proposition 12.1.3, we know that for any Funk function P on $T\Omega$, the spray $\tilde{\mathbf{G}} := y^i \frac{\partial}{\partial x^i} - 2P \frac{\partial}{\partial y^i}$ is R-flat. By [GrMu], $\tilde{\mathbf{G}}$ is locally induced by a (possibly not positive definite) Finsler metric $\tilde{F}$ on Ω. According to Proposition 12.2.7, $\tilde{F}$ is a solution of (12.47). Therefore there exists non-trivial pointwise projectively flat R-flat Finsler metrics. This is just Theorem 10.3.5. See further discussion in Section 12.4 on the inverse problems.

We continue to derive conditions equivalent to (12.44) or (12.47). Let $\tilde{F}$ be a positive definite Finsler metric on a positive definite Finsler space (M, F). Assume $\tilde{F}$ satisfies (12.47) for some positively homogeneous function P on TM. This implies

$$\tilde{F}_{;i\cdot j} = \tilde{F}_{;j\cdot i}. \tag{12.48}$$

Contracting (12.48) with y^i gives

$$\tilde{F}_{;i\cdot j} y^i = \tilde{F}_{;j\cdot i} y^i = \tilde{F}_{;j}.$$

We obtain another equivalent version of Rapcsák's Theorem.

Proposition 12.2.9 *Let (M, F) be a Finsler space and $\tilde{F}$ another Finsler metric on M. $\tilde{F}$ is pointwise projective to F if and only if $\tilde{F}$ satisfies*

$$\tilde{F}_{;i\cdot j} = \tilde{F}_{;j\cdot i}. \tag{12.49}$$

We still assume that $\tilde{F}$ is pointwise projective to F. As a scalar function on TM, the covariant derivatives of $\tilde{F}$ with respect to F satisfy the following Ricci identities

$$\tilde{F}_{;k\cdot l} = \tilde{F}_{\cdot l;k}, \tag{12.50}$$

$$\tilde{F}_{\cdot i\cdot j;k} = \tilde{F}_{\cdot i;k\cdot j} + \tilde{F}_m B^m_{ijk}. \tag{12.51}$$

Here B^m_{ijk} denote the Berwald curvature coefficients. Differentiating (12.44) with respect to y^j yields

$$\tilde{F}_{;k\cdot i\cdot j}y^k + \tilde{F}_{;j\cdot i} = \tilde{F}_{;i\cdot j}. \tag{12.52}$$

With both (12.50) and (12.48), we obtain

$$\tilde{F}_{i;k\cdot j}y^k = \tilde{F}_{;k\cdot i\cdot j}y^k = 0. \tag{12.53}$$

It follows from (12.51) and (12.53) that

$$\tilde{F}_{\cdot i\cdot j;k}y^k = \tilde{F}_{\cdot i;k\cdot j}y^k + \tilde{F}_m B^m_{ijk}y^k = 0. \tag{12.54}$$

Thus (12.54) is a necessary condition for $\tilde{F}$ to be pointwise projective to F. This necessary condition was obtained by M. Matsumoto [Ma9] in 1993.

Corollary 12.2.10 ([Ham]) *Let $\tilde{F} = \tilde{F}(x,y)$ be a positive definite Finsler metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$. Then $\tilde{F}$ is pointwise projective to the standard Euclidean metric if and only if one of the following conditions is satisfied.*

$$\tilde{F}_{x^k y^l}y^k = \tilde{F}_{x^l}, \tag{12.55}$$

$$\tilde{F}_{x^k y^l} = \tilde{F}_{x^l y^k}. \tag{12.56}$$

Any of the above conditions implies the following

$$\tilde{F}_{y^i y^j x^k}y^k = 0. \tag{12.57}$$

If F_1, F_2 are two pointwise projectively related positive definite Finsler metrics and α_1, α_2 two positive numbers, then

$$F := \alpha_1 F_1 + \alpha_2 F_2$$

also satisfies (12.44). Thus F is pointwise projective to F_1 and F_2, provided that F is still a positive definite Finsler metric.

A positive definite Finsler metric F on an open subset $\mathcal{U} \subset \mathrm{R}^n$ is called a *projective Finsler metric* if it is pointwise projective to the Euclidean metric. Projective Finsler metrics are, of course, pointwise projectively flat. Conversely, every locally projectively flat Finsler metric is locally isometric to a projective Finsler metric. There are lots of projective Finsler metrics. Two dimensional projective Finsler metrics are studied in [Bw5][Ham][Ma10][Ma11]. In [Bw5], L. Berwald gives explicit formulas for all two-dimensional projectively flat Finsler metrics with constant main scalar. See [Alv1][Alv2][AlGeSm] for some constructions in higher dimensions.

Let F denote the Funk metric on a strongly convex domain Ω in R^n. By Lemma 2.3.1, F satisfies

$$F_{x^k} = FF_{y^k}. \tag{12.58}$$

It follows from (12.58) that

$$F_{x^k y^l} y^k = F_{x^l}.$$

By Corollary 12.2.10, we conclude that F is pointwise projective to the Euclidean metric. More precisely, the spray coefficients of F are given by $G^i(y) = P(y)y^i$ where

$$P = \frac{F_{x^k} y^k}{2F} = \frac{1}{2}F.$$

Thus

$$\Xi = P^2 - P_{x^k} y^k = -\frac{1}{4}F^2.$$

By Theorem 12.2.4, F is of constant curvature $\mathbf{K} = -1/4$.

Let $\bar{F}(y) := F(-y)$ denote the *reverse* of F. By the same argument, one can easily show that $\bar{F}$ is also pointwise projective to the Euclidean metric. Further, $\bar{F}$ is also of constant curvature $\mathbf{K} = -\frac{1}{4}$. Since both $\bar{F}$ and F are pointwise projective to the Euclidean metric, so is $\tilde{F} := \frac{1}{2}(\bar{F} + F)$. The spray coefficients $\tilde{G}^i = \tilde{P}y^i$ of $\tilde{F}$ is given by

$$\tilde{P} = \frac{\tilde{F}_{x_k} y^k}{2\tilde{F}} = \frac{1}{2}(F - \bar{F}).$$

Observe that

$$\tilde{P}_{x^k} y^k = \frac{1}{4}\Big((F^2)_{y^k} + (\bar{F}^2)_{y^k}\Big)y^k = \frac{1}{2}\Big(F^2 + \bar{F}^2\Big).$$

This gives

$$\Xi = \tilde{P}^2 - \tilde{P}_{;k} y^k = -\tilde{F}^2.$$

By Theorem 12.2.4, we conclude that the Klein metric $\tilde{F}$ has constant curvature $\mathbf{K} = -1$. We summarize as follow.

Theorem 12.2.11 ([Fu1][Bw3][Ok]) *Let Ω be a strongly convex domain in R^n. Let F denote the Funk metric on Ω, $\bar{F}$ the reverse of F and $\tilde{F} := \frac{1}{2}(\bar{F} + F)$ the Klein metric on Ω. F, $\bar{F}$ and $\tilde{F}$ are pointwise projective to the Euclidean metric on R^n. Both F and $\bar{F}$ have constant curvature $\mathbf{K} = -1/4$, while $\tilde{F}$ has constant curvature $\mathbf{K} = -1$.*

12.3 Projectively Related Einstein Metrics

In this section we will continue to study projectively related Finsler metrics. We will show that if two Finsler metrics are pointwise projectively related, then along each geodesic, their Ricci scalars are related by a simple equation. This leads to some important results on projectively related Einstein metrics.

Let F and $\tilde{F}$ be positive definite Finsler metrics on a manifold M. Put

$$\mathbf{Ric}(y) = (n-1)\lambda(y)\, F^2(y), \qquad \widetilde{\mathbf{Ric}}(y) = (n-1)\tilde{\lambda}(y)\, \tilde{F}^2(y),$$

where $\lambda(y)$ and $\tilde{\lambda}(y)$ are scalar functions on TM. Suppose that F and $\tilde{F}$ are pointwise projectively related. Then (12.38) holds, i.e.,

$$\tilde{\lambda}(y)\tilde{F}^2(y) - \lambda(y)F^2(y) = \Big[\frac{\tilde{F}_{;k}y^k}{2\tilde{F}}\Big]^2 - \Big[\frac{\tilde{F}_{;k}y^k}{2\tilde{F}}\Big]_{;l}y^l. \tag{12.59}$$

Let $c(t)$ be a unit speed geodesic of F and $\tilde{c}(\tilde{t})$ be a unit speed geodesic of $\tilde{F}$ such that $c = \tilde{c}$ as a set of points, namely,

$$c(t) = \tilde{c}(\tilde{t}), \qquad \frac{d\tilde{t}}{dt} > 0.$$

Let

$$\lambda(t) := \lambda(\dot{c}(t)), \qquad \tilde{\lambda}(\tilde{t}) := \tilde{\lambda}(\dot{\tilde{c}}(\tilde{t})).$$

$$f(t) := \frac{1}{\sqrt{\tilde{F}(t)}}, \qquad \tilde{f}(\tilde{t}) = \frac{1}{\sqrt{F(\tilde{t})}}.$$

It follows from (12.59) that

$$f''(t) + \lambda(t)f(t) = \frac{\tilde{\lambda}(\tilde{t})}{f^3(t)}. \tag{12.60}$$

$$\tilde{f}''(\tilde{t}) + \tilde{\lambda}(\tilde{t})\tilde{f}(\tilde{t}) = \frac{\lambda(t)}{\tilde{f}^3(\tilde{t})}. \tag{12.61}$$

Equations (12.60) and (12.61) can also be expressed as follows.

$$-\frac{1}{2}\Big(\frac{d\tilde{t}}{dt}\Big)\frac{d^3\tilde{t}}{dt^3} + \frac{3}{4}\Big(\frac{d^2\tilde{t}}{dt^2}\Big)^2 + \lambda(t)\Big(\frac{d\tilde{t}}{dt}\Big)^2 = \tilde{\lambda}(\tilde{t})\Big(\frac{d\tilde{t}}{dt}\Big)^4. \tag{12.62}$$

$$-\frac{1}{2}\Big(\frac{dt}{d\tilde{t}}\Big)\frac{d^3t}{d\tilde{t}^3} + \frac{3}{4}\Big(\frac{d^2t}{d\tilde{t}^2}\Big)^2 + \tilde{\lambda}(\tilde{t})\Big(\frac{dt}{d\tilde{t}}\Big)^2 = \lambda(t)\Big(\frac{dt}{d\tilde{t}}\Big)^4. \tag{12.63}$$

Note that (12.63) is the inverse of (12.62) and vice versa.

From now on, we assume that F and $\tilde{F}$ are Einstein metrics, i.e., λ and $\tilde{\lambda}$ are constants. In this case, (12.60) is solvable, namely, the general solution can be expressed in terms of elementary functions. The solution of (12.60) with

$$f(0) = a > 0, \qquad f'(0) = b \neq 0$$

is determined by

$$\int_a^{f(t)} \frac{s}{\sqrt{-\lambda s^4 + 2Cs^2 - \tilde{\lambda}}} ds = \pm t, \tag{12.64}$$

where the sign $\pm$ in (12.64) is same as that of $f'(0) = b$ and

$$C := \frac{1}{2}\Big(\lambda a^2 + \tilde{\lambda}/a^2 + b^2\Big).$$

When $b=0$, the solution can be obtained by letting $b\to 0$.

Note that

$$-\lambda\Big(a^2 - C/\lambda\Big)^2 + C^2/\lambda - \tilde{\lambda} = (ab)^2 > 0, \quad \text{if } \lambda\neq 0 \tag{12.65}$$

and

$$-\lambda a^4 + 2Ca^2 - \tilde{\lambda} = (ab)^2 > 0. \tag{12.66}$$

Thus the integrand in (12.64) is defined for s close to a and the maximal solution $f(t)>0$ is defined on an interval I containing $s=0$.

By evaluating the integral in (12.64), we can prove the following

Theorem 12.3.1 ([Sh7]) *Let F and $\tilde{F}$ be pointwise projectively related positive definite Finsler metrics on a compact n-manifold M. Suppose that both F and $\tilde{F}$ are Einstein metrics with*

$$\mathbf{Ric} = (n-1)\lambda F^2, \qquad \widetilde{\mathbf{Ric}} = (n-1)\tilde{\lambda}\tilde{F}^2,$$

where $\lambda, \tilde{\lambda}\in\{-1,0,1\}$. Then λ and $\tilde{\lambda}$ have the same sign. More details are given below.

(i) If $\lambda = 1 = \tilde{\lambda}$, then along any unit speed geodesic $c(t)$ of F

$$\tilde{F}(\dot{c}(t)) = \frac{2}{\Big(a^2 - 1/a^2 - b^2\Big)\cos(2t) + 2ab\sin(2t) + \Big(a^2+1/a^2+b^2\Big)}, \tag{12.67}$$

where $a>0$ and $-\infty<b<\infty$ are constants. Thus, for any unit speed geodesic segment c of F with length of π, it is also a geodesic segment of $\tilde{F}$ with length of π.

(ii) If $\lambda = 0 = \tilde{\lambda}$, then along any geodesic $c(t)$ of F or $\tilde{F}$,

$$\frac{F(\dot{c}(t))}{\tilde{F}(\dot{c}(t))} = constant. \tag{12.68}$$

(iii) If $\lambda = -1 = \tilde{\lambda}$, then $\tilde{F} = F$.

To prove Theorem 12.3.1, we divide the problem into several cases.

Case 1: $\lambda = 1$. From (12.64), we obtain

$$f(t) = \sqrt{(a^2 - C)\cos(2t) + ab\sin(2t) + C}. \tag{12.69}$$

Using (12.65) we can rewrite (12.69) in the following form

$$f(t) = \sqrt{\sqrt{C^2-\tilde{\lambda}}\;\sin\Big[\sin^{-1}\Big(\frac{a^2-C}{\sqrt{C^2-\tilde{\lambda}}}\Big)\pm 2t\Big] + C}, \tag{12.70}$$

where the sign $\pm$ in (12.70) is same as that of $f'(0) = b$ when $b \neq 0$. Otherwise, the sign can be chosen arbitrarily.

Case 2: $\lambda = 0$. From (12.64) we obtain

$$f(t) = \sqrt{\Big(a + bt\Big)^2 + \tilde{\lambda}\Big(\frac{t}{a}\Big)^2}. \tag{12.71}$$

Case 3: $\lambda = -1$. From (12.64), we obtain

$$f(t) = \sqrt{(a^2 + C)\cosh(2t) + ab\sinh(2t) - C}. \tag{12.72}$$

Using (12.65) we can rewrite (12.72) as follows

$$f(t) = \begin{cases} \sqrt{\sqrt{C^2 + \tilde{\lambda}}\ \cosh\Big[\cosh^{-1}\Big(\frac{a^2+C}{\sqrt{C^2+\tilde{\lambda}}}\Big) \pm 2t\Big] - C} & \text{if } C^2 + \tilde{\lambda} > 0 \\ \sqrt{e^{\pm 2t}\Big(a^2 + C\Big) - C} & \text{if } C^2 + \tilde{\lambda} = 0 \\ \sqrt{\sqrt{-C^2 - \tilde{\lambda}}\ \sinh\Big[\sinh^{-1}\Big(\frac{a^2+C}{\sqrt{-C^2-\tilde{\lambda}}}\Big) \pm 2t\Big] - C} & \text{if } C^2 + \tilde{\lambda} < 0 \end{cases} \tag{12.73}$$

The sign $\pm$ in (12.73) is same as that of $f'(0) = b \neq 0$.

The proof of Theorem 12.3.1 follows from the above formulas (12.69)-(12.73) for $f(t) = [\tilde{F}(\dot{c}(t))]^{-1/2}$. First, we know that any Finsler metric on a closed manifold is complete. Thus $f(t) > 0$ must be defined on $(-\infty, \infty)$. The completeness of $\tilde{F}$ implies

$$\int_{-\infty}^{0} \frac{1}{f(t)^2} dt = \infty = \int_{0}^{\infty} \frac{1}{f(t)^2} dt.$$

Then the Einstein constants must have the same sign. The reader can easily prove Theorem 12.3.1 (i)-(iii). Q.E.D.

The non-compact case is more complicated. We list some theorems below.

Theorem 12.3.2 ([Sh7]) *Let F and $\tilde{F}$ be pointwise projectively related positive definite Finsler metrics on a non-compact n-manifold M. Suppose that both F and $\tilde{F}$ are Ricci-flat. Then along any unit speed geodesic $c(t)$ of F,*

$$\tilde{F}(\dot{c}(t)) = \frac{1}{(a + bt)^2}, \tag{12.74}$$

where $a > 0$ and $-\infty < b < \infty$ are constants. Thus F is complete if and only if $\tilde{F}$ is complete. In this case, along any geodesic $c(t)$ of F or $\tilde{F}$,

$$\frac{F(\dot{c}(t))}{\tilde{F}(\dot{c}(t))} = constant.$$

We actually prove that if c is a unit speed geodesic of F, along which $F/\tilde{F} \neq$ *constant*, then c can not be defined on $(-\infty, \infty)$. If c is defined on $[0, \infty)$ or $(-\infty, 0]$, then it has finite $\tilde{F}$-length. This suggests that there might be a positively complete (Ricci-)flat metric and a negatively complete (Ricci-)flat metric which are pointwise projective to each other. Such examples have not been found yet.

Theorem 12.3.3 ([Sh7]) *Let F and $\tilde{F}$ be pointwise projectively related positive definite Finsler metrics on a non-compact n-manifold M. Suppose that F and $\tilde{F}$ are Einstein metrics with*

$$\mathbf{Ric} = -(n-1)\, F^2, \qquad \widetilde{\mathbf{Ric}} = -(n-1)\, \tilde{F}^2.$$

Then along any geodesic $c(t)$ of F,

$$\tilde{F}(\dot{c}(t)) = \frac{2}{\Big(-1/a^2 + a^2 + b^2\Big)\cosh(2t) + 2ab\sinh(2t) - \Big(-1/a^2 - a^2 + b^2\Big)}, \tag{12.75}$$

where $a > 0$ and $-\infty < b < \infty$ are constants. If both F and $\tilde{F}$ are complete, then $F = \tilde{F}$.

Corollary 12.3.4 *Let F be a Finsler metric on a strongly convex domain Ω in R^n. Suppose that the following hold*

(a) F is complete;

(b) the geodesics of F are straight lines;

(c) F has constant curvature $\lambda = -1$.

Then F is the Klein metric on Ω.

12.4 Inverse Problems

A spray $\mathbf{G}$ on a manifold M is said to be *locally Finslerian* if at any point $x \in M$, there is a neighborhood $\mathcal{U}_x$ in which $\mathbf{G}$ is induced by a Finsler metric F. G is said to be *globally Finslerian*, if it is induced by a Finsler metric on the whole manifold.

Problem 1: *Given a spray, is it locally Finslerian? If it is locally Finslerian, is it globally Finslerian ?*

If a two-dimensional spray $\mathbf{G}$ is induced by a Finsler metric L, then the Riemann curvature takes the following form (8.62), namely,

$$\mathbf{R}_y(u) = \mathbf{K}(y)\Big\{L(y)u - g_y(y,u)\,y\Big\}$$

where $\mathbf{K}(y) = \mathbf{Ric}(y)/L(y)$. Thus $\mathbf{R} = 0$ if and only if $\mathbf{Ric} = 0$. This implies that any two-dimensional spray $\mathbf{G}$ with $\mathbf{Ric} = 0$ and $\mathbf{R} \neq 0$ is not Finslerian.

A spray on a manifold is said to be *locally projectively Finslerian* if at each point, there is a neighborhood in which it has the same geodesics as a Finsler metric. A spray is said to be *globally projectively Finslerian* if it has the same geodesics as a Finsler metric on the whole manifold. The following is the famous Hilbert's Fourth Problem.

Hilbert's Fourth Problem: *Given a domain $\Omega \subset \mathrm{R}^n$, determine all Finsler metrics on Ω whose geodesics are straight lines.*

There are a number of papers on the Hilbert Fourth Problem. See [Alv2] [Ale] [Dar] [Po] [Sz3] [Za],etc. In Hilbert's Fourth Problem, the base metric is the standard Euclidean metric on R^n. Thus if a Finsler metric is pointwise projective to the Euclidean metric, then it must be of scalar curvature (Theorem 12.2.4). It turns out that there are many Finsler metrics on a domain in R^n which are pointwise projective to the Euclidean metric. In particular, the Funk metrics and the Klein metrics are all pointwise projective to the Euclidean. Some Randers metrics are also pointwise projective to the Euclidean metric. For a Riemannian metric $\alpha(y) = \sqrt{g(y,y)}$ and 1-form β on a domain $\Omega \subset \mathrm{R}^n$, if α is pointwise projective to the Euclidean metric and $d\beta = 0$, then the Randers metric $F(y) := \alpha(y) + \beta(y)$ is also pointwise projective to the Euclidean metric. The following problem is natural.

Problem 2: *Given a spray, is it locally projectively Finslerian ? If it is locally projectively Finslerian, is it globally projectively Finslerian ?*

Before we discuss Problem 2, let us take a look at the following

Example 12.4.1 Consider the following spray on R^2

$$\mathbf{G} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - \frac{v}{r}\sqrt{u^2+v^2}\frac{\partial}{\partial u} + \frac{u}{r}\sqrt{u^2+v^2}\frac{\partial}{\partial v}, \tag{12.76}$$

where $r > 0$. By Example 4.1.3, we know that the geodesics of $\mathbf{G}$ are given by

$$\begin{aligned} x(t) &= r\cos\left(\frac{\lambda}{r}t + \theta\right) - r\cos\theta + x(0), \\ y(t) &= r\sin\left(\frac{\lambda}{r}t + \theta\right) - r\sin\theta + y(0). \end{aligned}$$

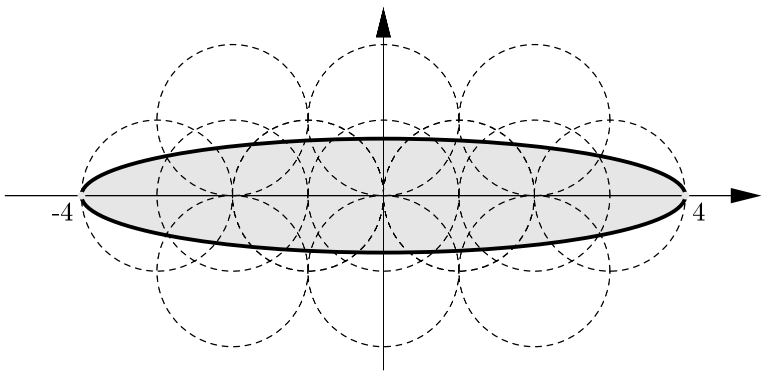
Thus the geodesics of $\mathbf{G}$ are circles of radius r. We are going to show that $\mathbf{G}$ is locally pointwise projective to a Randers metric everywhere. For any point $(x_o, y_o) \in \mathrm{R}^2$, define

$$F := \sqrt{u^2+v^2} + \frac{1-k}{r}(y-y_o)\,u - \frac{k}{r}(x-x_o)\,v, \tag{12.77}$$

where k is an arbitrary constant. F is indeed a Randers metric on an open domain Ω in R^2, where Ω is given by

$$\Big(\frac{k}{r}\Big)^2(x-x_o)^2 + \Big(\frac{1-k}{r}\Big)^2(y-y_o)^2 < 1.$$

When $k = 0, 1$, Ω is an open subset between two lines. When $k \neq 0, 1$, Ω is enclosed by an ellipse.



$k = 0.25,\ r = 1,\ (x_o, y_o) = (0, 0)$

To prove that F is pointwise projective to $\mathbf{G}$ on Ω, we need to compute the spray of F,

$$\tilde{\mathbf{G}} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - 2\tilde{G}\frac{\partial}{\partial u} - 2\tilde{H}\frac{\partial}{\partial v}.$$

According to Example 5.2.2, the spray coefficients $\tilde{G}$ and $\tilde{H}$ of F are given by

$$\begin{aligned}\tilde{G} &= Pu + \frac{v}{2r}\sqrt{u^2+v^2}\\ \tilde{H} &= Pv - \frac{u}{2r}\sqrt{u^2+v^2},\end{aligned}$$

where

$$P = \frac{1}{2F}\Big\{\frac{1-2k}{r}\,uv - \frac{\sqrt{u^2+v^2}}{r^2}\Big(k(x-x_o)u + (1-k)(y-y_o)v\Big)\Big\}.$$

Thus the spray $\tilde{\mathbf{G}}$ is pointwise projective to $\mathbf{G}$ on Ω. Note that when k changes, the domain Ω of F changes too. But the orbits of the geodesics remain unchanged.

We will show that the spray **G** is not globally projectively Finslerian. More precisely, it is not pointwise projective to any *positive definite* Finsler metric. See Example 14.1.1 below. ♯

Therefore, another problem arises: If a spray on a manifold is (locally) Finslerian, is it induced (locally) by a positive definite Finsler metric ?

In [Sz1], Z.I. Szabó studied the following inverse problem on affine sprays.

Problem 3: *Given an affine spray, under what sufficient and necessary conditions, the spray is locally Finslerian ?*

By definition, a spray is affine if the Berwald connection is an affine connection. A Finsler metric is Berwaldian if the induced spray is affine. Z. Szabó first proved that if an affine spray is induced by a positive definite Finsler metric, then it can be induced by a Riemannian metric. Based on this observation, he proved the following

Theorem 12.4.1 (Szabó) *Let* **G** *be an affine spray on a manifold* M *whose Riemann curvature* **R** *does not vanish everywhere. Then* **G** *is induced by a non-Riemannian positive definite Berwald metric if and only if the following two conditions are satisfied:*

(a) **G** *is induced by a Riemannian metric.*

(b) The Berwald connection of **G** *is either locally reducible or it is a locally irreducible, locally symmetric connection of rank* $k \geq 2$*.*

We remark that for a given affine spray on a manifold, it is hard to check whether or not the condition (a) in Theorem 12.4.1 is satisfied. Nevertheless, by Theorem 12.4.1, Szabó concludes that a positive definite Finsler metric inducing an affine spray must be one of the following four types:

(a) F is Riemannian;

(b) F is locally Minkowskian ;

(c) F is non-Riemannian and the Berwald connection D is locally irreducible, locally symmetric of rank $r \geq 2$;

(d) F is non-Riemannian and the Berwald connection D is locally reducible. In this case, the space can be locally decomposed to a Descartes product of Riemannian space, locally Minkowskian spaces and locally irreducible, locally symmetric non-Riemannian positive definite Berwald spaces of rank $r \geq 2$.

Szabó gave a full list of 56 kinds of irreducible and globally symmetric Berwald metrics.

As we know, affine sprays and isotropic sprays are two important classes of sprays. Thus it is natural to investigate the inverse problem on isotropic sprays.

Problem 4: *Given an isotropic spray, is it locally Finslerian ? If it is locally Finslerian, is it globally Finslerian ?*

The inverse problem on isotropic (semi)sprays has been studied by J. Grifone and Z. Muzsnay [GrMu]. The results of their study can be applied to construct Lagrange metrics and Finsler metrics of scalar curvature.

Now we make few remarks on the inverse problems on semisprays. By definition, a semispray $\mathcal{S}$ on $\mathcal{U} = (a,b)\times\Omega$ is *locally Lagrangian* if for any $(s,\eta)\in\mathcal{U}$, there exists an subinterval (a',b') of (a,b) and an open neighborhood Ω' of η such that $\mathcal{S}$ on $\mathcal{U}' = (a',b')\times\Omega'$ is induced by a Lagrange metric on $\mathcal{U}'$. See Chapter 3. We may ask the following questions.

Problem 5: *Given a semispray, is it locally Lagrangian ? If it is locally Lagrangian, is it globally Lagrangian ?*

Problems 2 is closely related to Problem 5.

Proposition 12.4.2 *Let $\mathcal{U} = (a,b)\times\Omega \subset \mathrm{R}^n$ be an open subset. Given a semispray $\mathcal{S} = \frac{\partial}{\partial s} + \xi^a\frac{\partial}{\partial \eta^a} + \Phi^a(s,\eta,\xi)\frac{\partial}{\partial \xi^a}$ and a spray $\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ on $\mathcal{U}$. Suppose that they are related by*

$$\Phi^a(s,\eta,\xi) = 2\xi^a G^1(s,\eta,1,\xi) - 2G^a(s,\eta,1,\xi), \qquad a = 2,\cdots,n. \tag{12.78}$$

$\mathcal{S}$ is locally induced by a Lagrange metric $\phi = \phi(s,\eta,\xi)$ on $\mathcal{U}$ if and only if $\mathbf{G}$ is locally projectively induced by a Finsler metric $L = L(x,y)$ on $\mathcal{U}$ in the form

$$L(x,y) = \Big[y^1\phi(s,\eta,\xi)\Big]^2, \tag{12.79}$$

where $s = x^1, \eta^a = x^a$ and $\xi^a = y^a/y^1$, $a = 2,\cdots,n$.

Proof: We first assume that $\mathcal{S}$ is locally induced by a Lagrange metric ϕ on $\mathcal{U}$. Define a Finsler metric L by (12.79). By Proposition 3.3.1, the spray $\tilde{\mathbf{G}} = y^i\frac{\partial}{\partial x^i} - 2\tilde{G}^i(x,y)\frac{\partial}{\partial y^i}$ of L is related to the semispray $\mathcal{S}$ by

$$\Phi^a(s,\eta,\xi) = 2\xi^a\tilde{G}^1(s,\eta,1,\xi) - 2\tilde{G}^a(s,\eta,1,\xi), \qquad a = 2,\cdots,n. \tag{12.80}$$

It follows from (12.78) and (12.80) that there is a function $P(x,y)$ such that $\tilde{G}^i(x,y) = G^i(x,y) + P(x,y)y^i$. Thus $\mathbf{G}$ is locally pointwise projective to $\tilde{\mathbf{G}}$. Namely, $\mathbf{G}$ is locally projectively induced by L.

Conversely, we assume that $\mathbf{G}$ is locally projectively induced by a Finsler metric L in the form (12.79) for some function ϕ (which must be a Lagrange metric). We claim that ϕ induces the semispray $\mathcal{S}$. Let $\tilde{\mathbf{G}} = y^i \frac{\partial}{\partial x^i} - 2\tilde{G}^i(x,y)\frac{\partial}{\partial y^i}$ denote the spray of L and $\tilde{\mathcal{S}} = \frac{\partial}{\partial s} + \xi^a \frac{\partial}{\partial \eta^a} + \tilde{\Phi}^a(s,\eta,\xi)\frac{\partial}{\partial \xi^a}$ the semispray of ϕ. By Proposition 3.3.1, $\tilde{\mathbf{G}}$ and $\tilde{\mathcal{S}}$ are related by

$$\tilde{\Phi}^a(s,\eta,\xi) = 2\xi^a \tilde{G}^1(s,\eta,1,\xi) - 2\tilde{G}^a(s,\eta,1,\xi), \qquad a = 2,\cdots,n. \tag{12.81}$$

On the other hand, by assumption, $\tilde{\mathbf{G}}$ is pointwise projective to $\mathbf{G}$. Namely, there is a function $P(x,y)$ such that $\tilde{G}^i(x,y) = G^i(x,y) + P(x,y)y^i$. This implies that

$$2\xi^a \tilde{G}^1(s,\eta,1,\xi) - 2\tilde{G}^a(s,\eta,1,\xi), = 2\xi^a G^1(s,\eta,1,\xi) - 2G^a(s,\eta,1,\xi). \tag{12.82}$$

Notice that the right hand side of (12.82) is equal to $\Phi^a(s,\eta,\xi)$ by (12.78). Thus $\tilde{\Phi}^a = \Phi^a$. This implies that $\mathcal{S} = \tilde{\mathcal{S}}$ is the semispray of ϕ. Q.E.D.

Theorem 12.4.3 (G. Darboux [Dar]) *Every two-dimensional spray is locally projectively Finslerian. Every one-dimensional semispray is locally Lagrangian.*

G. Darboux [Dar] proved that for any equation

$$\frac{d^2y}{dx^2} = \Phi\Big(x, y, \frac{dy}{dx}\Big), \tag{12.83}$$

there is a Lagrangian such that its Euler-Lagrange equation is (12.83). One can easily construct a (singular) Finsler metric in an open domain in R^2 such that its geodesics satisfy (12.83). Thus every two-dimensional spray is locally projectively Finslerian. M. Matsumoto provided a detailed formula of Finsler metrics for certain types of equations (see [Ma11]).

In higher dimensions, D.R. Davis [Dav1]-[Dav3] and D. Kosambi [Ko1][Ko2] first studied Problem 5. Finally, J. Douglas [Dg2] gave a complete solution to Problem 5 on semisprays in dimension two (i.e., semisprays on $(a,b)\times\Omega$, where $\dim\Omega = 2$). In [CSMBP], the authors gave a geometrical interpretation of the key parts of Douglas's paper [Dg2]. In [SCM], the authors made a general analysis on the integrability for the Helmholtz equations. They proved that a particular class of systems with n degree of freedom, corresponding to Class I of Douglas's classification in the two degrees of freedom case, is always Lagrangian. In [CPST], the authors extended other Douglas's results. In [GrMu], Grifone and Muzsnay solved the inverse problem for isotropic semisprays. This kinds of inverse problems form an important subject in mathematics.

Chapter 13

Douglas Curvature and Weyl Curvature

There are two important projective invariants of sprays and Finsler metrics. One is a non-Riemannian projective invariant constructed from the Berwald curvature. The other is a Riemannian projective invariant constructed from the Riemann curvature. In this chapter, we will discuss these two projective invariants.

13.1 Douglas Curvature of Sprays

Let $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}$ be a spray on a manifold M. Recall that for any vector $y \in T_xM \setminus \{0\}$, the Berwald curvature $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ is a trilinear form $\mathbf{B}_y(u, v, w) = B^i_{jkl}(y)u^jv^kw^l \frac{\partial}{\partial x^i}|_x$ defined by

$$B^i_{jkl}(y) := \frac{\partial^2 G^i}{\partial y^j \partial y^k \partial y^l}(y). \tag{13.1}$$

The mean Berwald curvature $\mathbf{E}_y : T_xM \times T_xM \to \mathrm{R}$ is a bilinear form $\mathbf{E}_y(u, v) = E_{ij}(y)u^iv^j$ defined by

$$E_{ij}(y) := \frac{1}{2} B^m_{ijm}(y). \tag{13.2}$$

Based on the Berwald curvature, J. Douglas [Dg1] introduced a new quantity $\mathbf{D}_y : T_xM \times T_xM \times T_xM \to T_xM$ which is a trilinear form $\mathbf{D}_y(u, v, w) = D^i_{jkl}(y)u^jv^kw^l \frac{\partial}{\partial x^i}|_x$ defined by

$$D^i_{jkl} := B^i_{jkl} - \frac{2}{n+1}\Big[E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + \frac{\partial E_{jk}}{\partial y^l} y^i\Big] \tag{13.3}$$

We call $\mathbf{D} := \{\mathbf{D}_y\}_{y \in TM\setminus\{0\}}$ the *Douglas curvature*. See also [Bw8].

Let

$$\Pi^i = G^i - \frac{1}{n+1} N^m_m(y) y^i .$$

$\Pi = y^i \frac{\partial}{\partial x^i} - 2\Pi^i(y)\frac{\partial}{\partial y^i}$ is the local projective spray associated with $\mathbf{G}$. Π depends only on the pointwise projective class of $\mathbf{G}$. Thus any (local) invariant defined by Π is a (local) projective invariant. Observe that

$$D^i_{jkl} = B^i_{jkl} - \frac{1}{n+1}\Big(N^m_m y^i\Big)_{y^j y^k y^l} \tag{13.4}$$

$$= \frac{\partial^3 \Pi^i}{\partial y^j \partial y^k \partial y^l}(y) \tag{13.5}$$

Thus $\mathbf{D}$ is a projective invariant. It follows from (13.5) that $\mathbf{D}_y$ is a symmetric trilinear form. Moreover,

$$\mathbf{D}_y(y, v, w) = 0, \qquad \mathrm{tr}\mathbf{D}_y(v, w) = 0. \tag{13.6}$$

Here $\mathrm{tr}\mathbf{D}_y(v,w) := \frac{1}{2} D^m_{jkm}(y) v^j w^k$ denotes the trace of $\mathbf{D}_y$.

The following lemma is trivial.

Lemma 13.1.1 $\mathbf{E} = 0$ *if and only if* $\mathbf{D} = \mathbf{B}$.

Proof: Suppose that $\mathbf{E} = 0$. It follows (13.3) that $\mathbf{D} = \mathbf{B}$. Suppose that $\mathbf{D} = \mathbf{B}$. By (13.6), we obtain

$$E_{ij} = \frac{1}{2} B^m_{ijm} = \frac{1}{2} D^m_{ijm} = 0.$$

Thus $\mathbf{E} = 0$. Q.E.D.

Proposition 13.1.2 *Every spray* $\mathbf{G}$ *on a manifold* M *is globally pointwise projective to a spray* $\tilde{\mathbf{G}}$ *with* $\tilde{\mathbf{B}} = \mathbf{D}$ *(equivalently,* $\tilde{\mathbf{E}} = 0$*). Hence* $\mathbf{G}$ *is globally pointwise projective to an affine spray* $\tilde{\mathbf{G}}$ *if and only if* $\mathbf{D} = 0$.

Proof: Take an arbitrary volume form $d\mu$ on M. Let $\mathbf{S}$ denote the S-curvature of $(\mathbf{G}, d\mu)$. Define

$$\tilde{\mathbf{G}} := \mathbf{G} + \frac{2\mathbf{S}}{n+1}\,\mathbf{Y}. \tag{13.7}$$

By Proposition 5.2.1, we know that $\tilde{\mathbf{S}} = 0$. Hence $\tilde{\mathbf{E}} = 0$ by (6.13). It follows from Lemma 13.1.1 that $\tilde{\mathbf{B}} = \tilde{\mathbf{D}} = \mathbf{D}$. Q.E.D.

Proposition 13.1.2 is a strength of the Douglas theorem ([Dg1][Bw8]). Douglas only proved a local version.

13.2 Projectively Affine Sprays and Douglas Metrics

Proposition 13.1.2 suggests the following

Definition 13.2.1 ([Dg1]) A spray is *projectively affine* if $\mathbf{D} = 0$. A Finsler metric is called a *Douglas metric* if its spray is projectively affine.

We are going to study projectively affine sprays and Douglas metrics. First of all, let us take a look at the following

Example 13.2.1 Consider a generalized Randers metric $\tilde{F} = F + \beta$ on a manifold M, where $F = F(y)$ is a Finsler metric on M and $\beta = b_i(x)y^i$ is a 1-form on M such that $\sup_{F(y)=1} |\beta(y)| < 1$. By (12.41), the spray coefficients of F and $\tilde{F}$ are related by

$$\tilde{G}^i = G^i + Py^i + Q^i, \tag{13.8}$$

where $P = \dfrac{\tilde{F}_{;k}y^k}{2\tilde{F}}$ and

$$Q^i = \frac{\tilde{F}}{2}\tilde{g}^{il}\Big\{\tilde{F}_{;k\cdot l}y^k - \tilde{F}_{;l}\Big\}. \tag{13.9}$$

Since $F_{;k} = 0$, we obtain

$$\tilde{F}_{;k\cdot l}y^k - \tilde{F}_{;l} = \beta_{;k\cdot l}y^k - \beta_{;l} = (b_{l;k} - b_{k;l})y^k.$$

This gives

$$Q^i = \frac{\tilde{F}}{2}\tilde{g}^{il}(b_{l;k} - b_{k;l})y^k.$$

Thus if β is close, i.e., $b_{r;s} - b_{s;r} = 0$, then $\tilde{\mathbf{D}} = \mathbf{D}$. In this case, F is a Douglas metric if and only if $\tilde{F}$ is a Douglas metric. This generalizes a result of S. Bácsó and M. Matsumoto [BaMa2]. They proved that for a Randers metric $F = \alpha+\beta$, F is a Douglas metric if and only if β is a close 1-form. ♯

One of the important problems in Finsler geometry is to describe Landsberg metrics which are not Berwaldian. So far, no explicit examples have been found yet. The following result shows that such examples do not exist among Douglas metrics.

Proposition 13.2.2 ([BaIlKi][BaMa1]) *For a Douglas metric L on a manifold M, if $\mathbf{L} = 0$, then $\mathbf{B} = 0$.*

Proof: We will sketch a proof for a positive definite Douglas metric $F(y) := \sqrt{L(y)}$. It follows from (13.3) that

$$B^l_{ijk} = \frac{2}{n+1}\Big\{E_{ij}\delta^l_k + E_{ik}\delta^l_j + E_{jk}\delta^l_i + \frac{\partial E_{ij}}{\partial y^k}y^l\Big\}, \tag{13.10}$$

where $E_{ij} := \frac{1}{2}B^m_{ijm}$. Let

$$h_{ij} := g_{ij} - \frac{1}{F^2} g_{ik}y^k g_{jl}y^l$$

and $h^i_j := g^{ik}h_{jk} = \delta^i_j - \frac{1}{F^2} g_{jl}y^l y^i$.

Assume that F is also a Landsberg metric, i.e., $L_{ijk} = -\frac{1}{2}y^s g_{sl}B^l_{ijk} = 0$. Contracting (13.10) with h^m_l yields

$$B^m_{ijk} = \frac{2}{n+1}\Big\{E_{ij}h^m_k + E_{ik}h^m_j + E_{jk}h^m_i\Big\}. \tag{13.11}$$

Substituting (13.11) into (10.12) yields

$$E_{ik}h_{jl} + E_{jk}h_{il} - E_{il}h_{jk} - E_{jl}h_{ik} = 0. \tag{13.12}$$

Contracting (13.12) with g^{ik} yields,

$$E_{jl} = \frac{\rho}{n-1}h_{jl}, \tag{13.13}$$

where $\rho := g^{ij}E_{ij}$. Plugging (13.13) into (13.11) gives

$$B^m_{ijk} = \frac{2\rho}{n^2-1}\Big\{h_{ik}h^m_j + h_{jk}h^m_i + h_{ij}h^m_k\Big\}. \tag{13.14}$$

Plugging (13.13) into (13.10) gives

$$\begin{aligned} B^m_{ijk} &= \frac{2\rho}{n^2-1}\Big\{h_{ik}h^m_j + h_{jk}h^m_i + h_{ij}h^m_k\Big\} \\ &\quad + \frac{2}{(n^2-1)F}\Big\{\frac{\partial}{\partial y^k}(F\rho)h_{ij} + 2\rho F C_{ijk}\Big\}y^m. \end{aligned} \tag{13.15}$$

Comparing (13.15) with (13.14), we obtain

$$h_{ij}\frac{\partial(\rho F)}{\partial y^k} = -2\rho F C_{ijk}. \tag{13.16}$$

Contracting (13.16) with g^{ij} yields

$$(n-1)\frac{\partial(\rho F)}{\partial y^k} = -2\rho F C_k, \tag{13.17}$$

where $C_k := g^{ij}C_{ijk}$. Substituting (13.17) into (13.16), we obtain

$$\Big\{C_k h_{ij} - (n-1)C_{ijk}\Big\}\rho F = 0. \tag{13.18}$$

Assume $n > 2$. Then (13.18) implies that either $C_k = 0$ or $\rho = 0$. If $C_k = 0$, then F is Riemannian by Deike's theorem [De][Bk2]. If $\rho = 0$, then $\mathbf{E} = 0$ by (13.13). This implies $\mathbf{B} = 0$ by (13.10). In either case, F is always a Berwald metric.

This fact is due to S. Bácsó and his collaborators [BaIlKi] [BaMa1] (claimed by H. Izumi [Iz] in 1984). In [BaIlKi], the Douglas metric is assumed to be positive definite as above. Later on, this condition was removed in [BaMa1]. The proof in dimension two is omitted. Q.E.D.

From Proposition 13.2.2, we immediately obtain the following

Corollary 13.2.3 *Let F be a positive definite R-quadratic Finsler metric on a compact manifold M. F is a Douglas metric if and only if it is a Berwald metric.*

Proof: Assume that F is a Douglas metric. By Theorem 10.3.2, F must be a Landsberg metric. Thus F must be a Berwald metric by Proposition 13.2.2. Q.E.D.

It is much easier to describe projectively affine sprays than Douglas metrics. Let $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ be a spray on an open subset $\mathcal{U} = I \times \Omega \subset \mathrm{R}^n$. Let

$$\Phi^a(s,\eta,\xi) := 2\xi^a G^1(s,\eta,1,\xi) - 2G^a(s,\eta,1,\xi), \qquad a = 2,\cdots,n, \tag{13.19}$$

where $\eta = (\eta^2,\cdots,\eta^n)$ and $\xi = (\xi^2,\cdots,\xi^n)$. Using Φ^a, we define a new spray $\hat{\mathbf{G}} = y^i \frac{\partial}{\partial x^i} - \hat{G}^i(x,y)\frac{\partial}{\partial y^i}$ by

$$\hat{G}^1 := 0, \qquad \hat{G}^a := -\frac{1}{2}y^1y^1\Phi^a\Big(x^1,\cdots,x^n;\frac{y^2}{y^1},\cdots,\frac{y^n}{y^1}\Big). \tag{13.20}$$

Note that $\mathbf{G}$ is pointwise projective to $\hat{\mathbf{G}}$. Using (13.4), we immediately obtain a local formula for the Douglas curvature.

$$\begin{aligned} D^a_{bcd} &= \frac{1}{2(n+1)}\Big(\Phi^e_{\xi^b\xi^c\xi^d\xi^e}\xi^a - (n+1)\Phi^a_{\xi^b\xi^c\xi^d}\Big) \\ &\quad +\delta^a_b\Phi^e_{\xi^c\xi^d\xi^e} + \delta^a_c\Phi^e_{\xi^b\xi^d\xi^e} + \delta^a_d\Phi^e_{\xi^b\xi^c\xi^e}, \end{aligned} \tag{13.21}$$

$$D^1_{bcd} = \frac{1}{2(n+1)}\Phi^e_{\xi^b\xi^c\xi^d\xi^e}. \tag{13.22}$$

All quantities on the left side are valued on $x = (s,\eta)$ and $y = (1,\xi)$. Rewrite (13.6) as follows

$$D^i_{jka}\xi^a + D^i_{jk1} = 0. \tag{13.23}$$

By the above formulas, we obtain the following

Proposition 13.2.4 ([Bw8]) *A spray $\mathbf{G}$ is projectively affine if and only if the set of functions Φ^a defined in (13.19) are in the following form*

$$\Phi^a = A^a + B^a_b\xi^b + C^a_{bc}\xi^b\xi^c + D_{bc}\xi^b\xi^c\xi^a, \tag{13.24}$$

where A^a, B^a_b, C^a_{bc} and D_{bc} are functions of (s,η).

Proof: It follows from (13.21)-(13.23) that if Φ^a are in the form (13.24), then all coefficients $D^i_{jkl} = 0$. Suppose $\mathbf{D} = 0$. By Proposition 13.1.2, $\mathbf{G}$ is pointwise projective to an affine spray $\tilde{\mathbf{G}} = y^i \frac{\partial}{\partial x^i} - 2\tilde{G}^i(y)\frac{\partial}{\partial y^i}$ with $\tilde{G}^i(y) = \frac{1}{2}\tilde{\Gamma}^i_{jk}(x)y^j y^k$. Thus

$$\begin{aligned}\Phi^a(s,\eta,\xi) &= 2\xi^a\tilde{G}^1(a,\eta,1,\xi) - 2\tilde{G}^a(s,\eta,1,\xi)\\ &= \xi^a\Big(\tilde{\Gamma}^1_{11} + 2\tilde{\Gamma}^1_{1b}\xi^b + \tilde{\Gamma}^1_{bc}\xi^b\xi^c\Big)\\ &\quad -\Big(\tilde{\Gamma}^a_{11} + 2\tilde{\Gamma}^a_{1b}\xi^b + \tilde{\Gamma}^a_{bc}\xi^b\xi^c\Big).\end{aligned}$$

Thus Φ^a must be in the form (13.24). Q.E.D.

Consider a Finsler metric on an open subset $\mathcal{U} = (a,b)\times\Omega \subset \mathrm{R}^n$ in the form

$$L(x,y) = \Big[y^1\phi\Big(s,\eta,\xi\Big)\Big]^2, \tag{13.25}$$

where $s = x^1, \eta^a = x^a$ and $\xi^a = y^a/y^1$, $a = 2,\cdots,n$. By Lemma 1.1.3, the function $\phi = \phi(s,\eta,\xi)$ must be a Lagrange metric on $\mathcal{U}$. By (3.28), ϕ defines a set of functions Φ^a

$$\Phi^a(s,\eta,\xi) = \frac{1}{2}h^{ab}\Big\{\phi_{\eta^b} - \phi_{s\xi^b} - \phi_{\eta^c\xi^b}\xi^c\Big\}, \tag{13.26}$$

where $h_{ab} := \frac{1}{2}\phi_{\xi^a\xi^b}(s,\eta,\xi)$. By Proposition 3.3.1, the spray coefficients $G^i = G^i(x,y)$ are related to $\Phi^a = \Phi^a(s,\eta,\xi)$ by (13.19). By Proposition 13.2.4, L is a Douglas metric if and only if Φ^a are in the form (13.24).

Example 13.2.2 Let $\mathcal{U} \subset \mathrm{R}^3$ be an open subset and (x,y,z,u,v,w) denote the standard coordinate system in $T\mathcal{U} = \mathcal{U}\times\mathrm{R}^3$. Consider a Finsler metric on $\mathcal{U}$ in the form

$$\begin{aligned}L &= u^2\Big[A(x,y,z)\Big(\frac{v}{u}\Big)^2 + B(x,y,z)\frac{v}{u}\frac{w}{u} + C(x,y,z)\Big(\frac{w}{u}\Big)^2\\ &\qquad P(x,y,z)\frac{v}{u} + Q(x,y,z)\frac{w}{u} + R(x,y,z)\Big]^2.\end{aligned} \tag{13.27}$$

By (13.26), ϕ defines two functions $\Phi := \Phi^2(x,y,z,\xi,\zeta)$ and $\Psi = \Phi^3(x,y,z,\xi,\zeta)$. A direct computation yields

$$\begin{aligned}\Phi &= \frac{1}{2B^2}\Big\{\Big(CA_y - 2BB_y + BA_z\Big)\xi^2 + +2\Big(CA_z - BC_y\Big)\xi\zeta\\ &\quad -\Big(CC_y - 2CB_z + BC_z\Big)\zeta^2 + \Big(2CA_x + BP_z - 2BB_x - BQ_y\Big)\xi\\ &\quad +\Big(2CB_x - CQ_y + CP_z - 2BC_x\Big)\zeta + CP_x - CR_y + BR_z - BQ_x\Big\}\\ \Psi &= -\frac{1}{2B}\Big\{A_y\xi^2 + 2A_z\xi\zeta + \Big(2B_z - C_y\Big)\zeta^2\\ &\quad +2A_x\xi + \Big(P_z - Q_y + 2B_x\Big)\zeta + P_x - R_y\Big\}.\end{aligned}$$

Thus L is a Douglas metric. ♯

New we consider a two-dimensional $\mathbf{G}$ on an open subset $\mathcal{U} = (a,b)\times(\alpha,\beta) \subset \mathrm{R}^2$. Express $\mathbf{G}$ as follows

$$\mathbf{G} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - 2G(x,y,u,v)\frac{\partial}{\partial u} - 2H(x,y,u,v)\frac{\partial}{\partial v}.$$

Let

$$\Phi(x,y,\xi) := 2\xi G(x,y,1,\xi) - 2H(x,y,1,\xi). \tag{13.28}$$

By (13.21)-(13.23), the Douglas curvature is given by

$$\mathbf{D}_{\mathbf{y}}(\mathbf{v},\mathbf{v},\mathbf{v}) = -\frac{1}{6}\Phi_{\xi\xi\xi\xi}u^{-2}(\xi\mu-\nu)^3\,\mathbf{y}, \tag{13.29}$$

where $\mathbf{y} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$ with $u>0$, $\mathbf{v} = \mu\frac{\partial}{\partial x} + \nu\frac{\partial}{\partial y}$ and $\xi = v/u$. We immediately obtain the following

Corollary 13.2.5 *Let $\mathbf{G}$ be a two-dimensional spray on an open subset $\mathcal{U} \subset \mathrm{R}^2$. $\mathbf{G}$ is projectively affine if and only if the function $\Phi = \Phi(x,y,\xi)$ defined in (13.28) is a polynomial of degree three or less in ξ,*

$$\Phi = A + B\xi + C\xi^2 + D\xi^3.$$

where A, B and C are some functions of $(x,y)\in\mathcal{U}$.

Consider the following Finsler metric on an open subset $\mathcal{U} = (a,b)\times\Omega \subset \mathrm{R}^2$,

$$L(x,y,u,v) = \Big[u\phi\Big(x,y,\frac{v}{u}\Big)\Big]^2. \tag{13.30}$$

The function $\phi = \phi(x,y,\xi)$ must be a Lagrange metric on $\mathcal{U}$. By (13.26), ϕ defines the following function

$$\Phi(x,y,\xi) := \frac{\phi_y - \phi_{x\xi} - \phi_{y\xi}\xi}{\phi_{\xi\xi}}.$$

Thus L is a Douglas metric if and only if $\Phi_{\xi\xi\xi\xi} = 0$, namely,

$$\frac{\phi_y - \phi_{x\xi} - \phi_{y\xi}\xi}{\phi_{\xi\xi}} = A + B\xi + C\xi^2 + D\xi^3, \tag{13.31}$$

where A, B and C are some functions of $(x,y)\in\mathcal{U}$.

There are two-dimensional R-flat sprays which are not projectively affine. See the spray defined in (8.52).

13.3 Weyl Curvature of Sprays

In the previous sections, we discussed an important projective invariant — the Douglas curvature. Since the Douglas curvature always vanishes for Riemannian spaces, it plays a role only outside the Riemannian world. In 1921, H. Weyl introduced a projective invariant for Riemannian metrics and affine sprays (affine connections) [We1]. Later on, J. Douglas [Dg1] extended Weyl's projective invariant to Finsler metrics and sprays. See also [Bw8].

Weyl's projective invariant is constructed from the Riemann curvature. Let

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(y) \frac{\partial}{\partial y^i}$$

be a spray on a manifold M. Let $\mathbf{R}_y(u) = R^i_{\ k}(y) u^k \frac{\partial}{\partial x^i}|_x$ denote the Riemann curvature of $\mathbf{G}$. The Ricci curvature $\mathbf{Ric}(y) = (n-1)R(y)$ is the trace of $\mathbf{R}_y$ expressed by

$$R(y) := \frac{1}{n-1} R^m_{\ m}(y). \tag{13.32}$$

For a vector $y \in T_xM \setminus \{0\}$, define $\mathbf{W}_y(u) = W^i_{\ k}(y) u^k \frac{\partial}{\partial x^i}|_x$ by

$$W^i_{\ k}(y) := A^i_{\ k} - \frac{1}{n+1} \frac{\partial A^m_{\ k}}{\partial y^m} y^i, \tag{13.33}$$

where

$$A^i_{\ k} := R^i_{\ k} - R \delta^i_k.$$

$\mathbf{W}_y : T_xM \to T_xM$ is a linear transformation satisfying

$$\mathbf{W}_y(y) = 0, \qquad \mathrm{tr}\mathbf{W}_y = 0.$$

Here $\mathrm{tr}\mathbf{W}_y := W^m_{\ m}(y)$. We call $\mathbf{W} := \{\mathbf{W}_y\}_{y \in TM\setminus\{0\}}$ the *Weyl curvature.*

To prove that $\mathbf{W}$ is a projective invariant, we consider another spray $\tilde{\mathbf{G}} = y^i \frac{\partial}{\partial x^i} - 2\tilde{G}^i(y) \frac{\partial}{\partial y^i}$, where

$$\tilde{G}^i(y) = G^i(y) + P(y)\, y^i.$$

It follows from (12.17) and (12.16) that

$$\tilde{R}^i_{\ k} = R^i_{\ k} + \Xi \delta^i_k + \tau_k y^i. \tag{13.34}$$

$$\tilde{R} = R + \Xi. \tag{13.35}$$

By (13.34) and (13.35), we obtain

$$\tilde{A}^i_{\ k} = A^i_{\ k} + \tau_k y^i. \tag{13.36}$$

Differentiating (13.36), we obtain another formula for τ_k

$$\frac{\partial \tilde{A}^m_{\ k}}{\partial y^m} = \frac{\partial A^m_{\ k}}{\partial y^m} + (n+1)\tau_k. \tag{13.37}$$

Plugging (13.37) back to (13.36) yields

$$\tilde{W}^i_k = \tilde{A}^i_k - \frac{1}{n+1}\frac{\partial \tilde{A}^m_k}{\partial y^m} y^i = A^i_k - \frac{1}{n+1}\frac{\partial A^m_k}{\partial y^m} y^i = W^i_k.$$

This shows that $\mathbf{W}$ is a projective invariant.

13.4 Isotropic Sprays and Finsler Metrics of Scalar Curvature

A spray is said to be isotropic if the Riemann curvature takes the following form

$$R^i_k = R\delta^i_k + \tau_k y^i \tag{13.38}$$

with $\tau_k y^k = -R$. We can express W^i_k as follows

$$W^i_k = R^i_k - R\delta^i_k - \zeta_k y^i, \tag{13.39}$$

where

$$\zeta_k := \frac{1}{n+1}\frac{\partial}{\partial y^m}\Big(R^m_k - R\delta^m_k\Big).$$

By the homogeneity of R^i_k, we obtain

$$\zeta_k y^k = \frac{1}{n+1}\frac{\partial}{\partial y^m}\Big(R^m_k - R\delta^m_k\Big) y^k = \frac{1}{n+1}\Big(- R^m_m - R\Big) = -R.$$

Thus if $W^i_k = 0$, then (13.38) holds by taking $\tau_k := \zeta_k$.

Suppose (13.38) holds for some τ_k. By the homogeneity of τ_k, we obtain

$$\zeta_k = \frac{1}{n+1}\frac{\partial}{\partial y^m}\Big(R^m_k - R\delta^m_k\Big) = \frac{1}{n+1}\frac{\partial}{\partial y^m}\Big(\tau_k y^m\Big) = \tau_k.$$

Thus

$$W^i_k = \Big[R^i_k - R\delta^i_k\Big] - \zeta_k y^i = \tau_k y^i - \zeta_k y^i = 0.$$

This gives the following

Proposition 13.4.1 *Let* $\mathbf{G}$ *be a spray on a manifold* M. *Then* $\mathbf{G}$ *is isotropic if and only if* $\mathbf{W} = 0$.

By Lemma 8.2.2, we know that a Finsler metric is of scalar curvature if and only if its spray is isotropic. By Proposition 13.4.1, we obtain the following

Corollary 13.4.2 ([Sz2][Ma3]) *A Finsler metric* L *is of scalar curvature if and only if its Weyl curvature* $\mathbf{W} = 0$.

Let $\tilde{L}$ and L be Finsler metrics on a manifold. Suppose that $\tilde{L}$ is pointwise projective to L. Since the Weyl curvature is a projective invariant, we immediately conclude that $\tilde{L}$ is of scalar curvature if and only if L is of scalar curvature. See [Sz2][MaWe][ShKi].

For a Riemannian space (M, g) of dimension ≥ 3, the following conditions are equivalent.

(a) $\mathbf{W} = 0$,

(b) g is of scalar curvature,

(c) g is of constant curvature,

(d) g is locally projectively flat.

It is much easier to describe isotropic sprays than Finsler metrics of scalar curvature. Let $\mathbf{G} := y^i \frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}$ be a spray on an open subset $\mathcal{U} = I \times \Omega \subset \mathrm{R}^n$. Set

$$\Phi^a(s, \eta, \xi) := 2\xi^a G^1(s, \eta, 1, \xi) - 2G^a(s, \eta, 1, \xi), \qquad a = 2, \cdots, n, \tag{13.40}$$

where $\eta = (\eta^2, \cdots, \eta^n)$ and $\xi = (\xi^2, \cdots, \xi^n)$. Then we define a new spray $\hat{\mathbf{G}} = y^i \frac{\partial}{\partial x^i} - 2\hat{G}^i(x, y)\frac{\partial}{\partial y^i}$ by

$$\hat{G}^1 := 0, \qquad \hat{G}^a := -\frac{1}{2} y^1 y^1 \Phi^a\Big(x^1, \cdots, x^n; \frac{y^2}{y^1}, \cdots, \frac{y^n}{y^1}\Big). \tag{13.41}$$

$\mathbf{G}$ is pointwise projective to $\hat{\mathbf{G}}$. Since the Weyl curvature is a projective invariant, we can compute the Weyl curvature $\mathbf{W}$ of $\mathbf{G}$ by computing that of $\hat{\mathbf{G}}$. An easy computation gives the formulas for the Riemann curvature $\hat{\mathbf{R}}$ of $\hat{\mathbf{G}}$.

$$\hat{R}^1_j = 0, \tag{13.42}$$

$$\hat{R}^a_b = -\Phi^a_{\eta^b} + \frac{1}{2}\Phi^a_{\eta^c\xi^b}\xi^c + \frac{1}{2}\Phi^a_{s\xi^b} + \frac{1}{2}\Phi^c\Phi^a_{\xi^b\xi^c} - \frac{1}{4}\Phi^a_{\xi^c}\Phi^c_{\xi^b} \tag{13.43}$$

$$\hat{R}^a_1 = -\hat{R}^a_b \xi^b, \tag{13.44}$$

where $\hat{R}^i_j$ are valued on $x = (s, \eta)$ and $y = (1, \xi)$. The Ricci scalar $\hat{R} = \frac{1}{n-1}\hat{R}^m_m$ is given by

$$\hat{R} = \frac{1}{n-1}\Big[-\Phi^a_{\eta^a} + \frac{1}{2}\Phi^a_{\eta^c\xi^a}\xi^c + \frac{1}{2}\Phi^a_{s\xi^a} + \frac{1}{2}\Phi^c\Phi^a_{\xi^a\xi^c} - \frac{1}{4}\Phi^a_{\xi^c}\Phi^c_{\xi^a}\Big]. \tag{13.45}$$

Let

$$\hat{A}^a_b := \hat{R}^a_b - \hat{R}\delta^a_b.$$

By (13.42), (13.33) simplifies to

$$W^1_b = -\frac{1}{n+1}\frac{\partial \hat{A}^c_b}{\partial \xi^c}, \tag{13.46}$$

$$W^a_b = \hat{A}^a_b - \frac{1}{n+1}\frac{\partial \hat{A}^c_b}{\partial \xi^c}\xi^a. \tag{13.47}$$

The rest of components of $\mathbf{W}$ can be obtained by

$$W^i_1 = -W^i_b \xi^b. \tag{13.48}$$

Plugging (13.44)-(13.45) into (13.46) and (13.47), we obtain the expressions for W^i_k in terms of Φ^a.

Proposition 13.4.3 ([Bw8]) *A spray* $\mathbf{G}$ *on an open subset* $\mathcal{U} = I \times \Omega \subset \mathrm{R}^n$ *is isotropic if and only if the Riemann curvature of the associated spray* $\hat{\mathbf{G}}$ *has the following form*

$$\hat{R}^a_b = \hat{R}\ \delta^a_b, \tag{13.49}$$

namely,

$$-\Phi^a_{\eta^b} + \frac{1}{2}\Phi^a_{\eta^c\xi^b}\xi^c + \frac{1}{2}\Phi^a_{s\xi^b} + \frac{1}{2}\Phi^c\Phi^a_{\xi^b\xi^c} - \frac{1}{4}\Phi^a_{\xi^c}\Phi^c_{\xi^b} = \hat{R}\ \delta^a_b. \tag{13.50}$$

Proof: It follows (13.46) and (13.47) that

$$W^a_b = \hat{A}^a_b + W^1_b\frac{y^a}{y^1}. \tag{13.51}$$

Clearly, $W^i_j = 0$ if and only if (13.49) holds. Q.E.D.

13.5 Projectively Flat Sprays and Finsler Metrics

A spray is *(locally) projectively R-flat* if it is (locally) pointwise projective to a R-flat spray. Locally projectively R-flat sprays are isotropic sprays. It is not clear if there exist isotropic sprays which are not projectively R-flat. A spray is said to be *locally projectively flat* if it is (locally) pointwise projective to a flat spray.

Theorem 13.5.1 ([Dg1]) *A spray* $\mathbf{G}$ *or a Finsler metric* L *on a manifold* M *(*$\dim M > 2$*) is locally projectively flat if and only if*

$$\mathbf{D} = 0, \qquad \mathbf{W} = 0. \tag{13.52}$$

An outline of the proof is given as follows. First, $\mathbf{D} = 0$ implies that the local projective spray Π associated with $\mathbf{G}$ (see (12.8)) is affine. In 1926, O. Veblen and J.M. Thomas [VeTh2] proved that any affine spray with $\mathbf{W} = 0$ is locally projectively flat. Thus $\mathbf{G}$ is locally projectively flat. This observation is made by J. Douglas [Dg1].

According to Propositions 13.2.4 and 13.4.3, when $\dim n \geq 3$, the local structure of projectively flat sprays is determined by a system of partial differential equations of some functions on the base manifold. To certain degree, we are satisfied with this result in Spray geometry. However, in Finsler geometry, we want to understand the local structure of projectively flat Finsler metrics, not only the induced sprays.

Below we are going to study the Berwald curvature of locally projectively flat sprays, then study the Landsberg curvature of locally projectively flat Finsler metrics.

First, by Proposition 13.4.1, a locally projectively flat spray is isotropic, namely, the Riemann curvature is in the following form

$$R^i_k = R\delta^i_k + \tau_k y^i$$

with $\tau_k y^k = -R$. In fact, τ_k are given by

$$\tau_k = \frac{1}{n+1}\frac{\partial}{\partial y^m}\Big(R^m_k - R\delta^m_k\Big).$$

Then (9.50) and (9.51) hold. On the other hand, the spray is affine. By (13.3), the Berwald curvature is in the following

$$B^i_{jkl} = \frac{2}{n+1}\Big[E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jk\cdot l}y^i\Big], \tag{13.53}$$

where $E_{jk\cdot l} = \frac{\partial E_{jk}}{\partial y^l}$. Taking a horizontal covariant derivative of B^i_{jkl} then contracting it with y^m yield

$$B^i_{jkl;m}y^m = \frac{2}{n+1}\Big[E_{jk;m}y^m\delta^i_l + E_{jl;m}y^m\delta^i_k + E_{kl;m}y^m\delta^i_j + E_{jk\cdot l;m}y^m y^i\Big] \tag{13.54}$$

We need the following Ricci identity for E_{ij},

$$E_{ij\cdot l;k} - E_{ij;k\cdot l} = E_{pj}B^p_{ikl} + E_{ip}B^p_{jkl}. \tag{13.55}$$

It follows from (13.55) that

$$E_{jk\cdot l;m}y^m = E_{jk;m\cdot l}y^m = \Big[E_{jk;m}y^m\Big]_{\cdot l} - E_{jk;l}. \tag{13.56}$$

Plugging (9.51) into (13.56) yields

$$E_{jk\cdot l;m}y^m = \frac{n+1}{6}\Big\{\tau_{m\cdot j\cdot k\cdot l}y^m + \tau_{l\cdot j\cdot k}\Big\} - E_{jk;l}. \tag{13.57}$$

Plugging (9.50), (9.51) and (13.57) into (13.54), we obtain the horizontal covariant derivative of $\mathbf{E}$

$$E_{jk;l} = \frac{n+1}{6}\Big\{\tau_{l\cdot j\cdot k} - R_{\cdot j\cdot k\cdot l}\Big\}. \tag{13.58}$$

Plugging (13.58) into (13.57) gives

$$E_{jk\cdot l;m}y^m = \frac{n+1}{6}\Big\{\tau_{m\cdot j\cdot k\cdot l}y^m + R_{\cdot j\cdot k\cdot l}\Big\}. \tag{13.59}$$

By (13.53), (13.55) and (13.57), we obtain

$$\begin{aligned} E_{jk\cdot l;m} &= \frac{n+1}{6}\Big\{\tau_{m\cdot j\cdot k\cdot l} - R_{\cdot j\cdot k\cdot l\cdot m}\Big\} \\ &\quad + \frac{4}{n+1}\Big\{E_{jm}E_{kl} + E_{jl}E_{km} + E_{jk}E_{lm}\Big\}. \end{aligned} \tag{13.60}$$

Using (13.58) and (13.60), we obtain the horizontal covariant derivatives of $\mathbf{B}$

$$\begin{aligned} B^i_{jkl;m} &= \frac{1}{3}\Big\{(\tau_{m\cdot j\cdot k} - R_{\cdot j\cdot k\cdot m})\delta^i_l + (\tau_{m\cdot j\cdot l} - R_{\cdot j\cdot l\cdot m})\delta^i_k \\ &\quad + (\tau_{m\cdot k\cdot l} - R_{\cdot k\cdot l\cdot m})\delta^i_j + \Big(\tau_{m\cdot j\cdot k\cdot l} - R_{\cdot j\cdot k\cdot l\cdot m}\Big)y^i\Big\} \\ &\quad + \frac{8}{(n+1)^2}\Big\{E_{jm}E_{kl} + E_{jl}E_{km} + E_{jk}E_{lm}\Big\}y^i. \end{aligned} \tag{13.61}$$

For a vector $y \in T_xM$, define

$$\begin{aligned} \bar{\mathbf{E}}_y(u,v,w) : &= E_{ij;k}(y)u^iv^jw^k, \\ \bar{\mathbf{B}}_y(u,v,w,z) : &= B^i_{jkl;m}(y)u^iv^jw^kz^l\frac{\partial}{\partial x^i}|_x, \end{aligned}$$

where $u = u^i\frac{\partial}{\partial x^i}|_x$ and etc. We obtain the following

Proposition 13.5.2 *For any locally projectively flat R-flat spray* $\mathbf{G}$ *on an n-dimensional manifold* M,

$$\bar{\mathbf{E}} = 0.$$

Moreover, $\bar{\mathbf{B}} = 0$ *if and only if* $\mathbf{G}$ *is locally flat.*

Proof: By assumption, $R = 0$ and $\tau_k = 0$. Then (13.58) and (13.61) simplify to

$$\begin{aligned} E_{jk;l} &= 0, \\ B^i_{jkl;m} &= \frac{8}{(n+1)^2}\Big\{E_{jm}E_{kl} + E_{jl}E_{km} + E_{jk}E_{lm}\Big\}y^i. \end{aligned}$$

Suppose $B^i_{jkl;m} = 0$. Then

$$E_{jm}E_{kl} + E_{jl}E_{km} + E_{jk}E_{lm} = 0. \tag{13.62}$$

For any vector $v = v^i \frac{\partial}{\partial x^i}|_x$,

$$3\Big(E_{ij}v^iv^j\Big)^2 = \Big\{E_{jm}E_{kl} + E_{jl}E_{km} + E_{jk}E_{lm}\Big\}v^jv^kv^lv^m = 0.$$

Thus $\mathbf{E} = 0$. Plugging $E_i = 0$ into (13.53) gives $\mathbf{B} = 0$. By Proposition 8.1.6, we conclude that $\mathbf{G}$ is locally flat. Q.E.D.

Now we study locally projectively flat Finsler metrics. Contracting (13.61) with $-\frac{1}{2}y^pg_{ip}$ gives

$$\begin{aligned} L_{jkl;m} \;=\;& -\frac{1}{6}\Big\{(\tau_{m\cdot j\cdot k} - R_{\cdot j\cdot k\cdot m})g_{lp}y^p + (\tau_{m\cdot j\cdot l} - R_{\cdot j\cdot l\cdot m})g_{kp}y^p \\ & +(\tau_{m\cdot k\cdot l} - R_{\cdot k\cdot l\cdot m})g_{jp}y^p + \Big(\tau_{m\cdot j\cdot k\cdot l} - R_{\cdot j\cdot k\cdot l\cdot m}\Big)L\Big\} \\ & -\frac{4}{(n+1)^2}\Big\{E_{jm}E_{kl} + E_{jl}E_{km} + E_{jk}E_{lm}\Big\}L. \end{aligned} \tag{13.63}$$

For a vector $y \in T_xM$, define

$$\bar{\mathbf{L}}_y(u, v, w, z) := \bar{L}_{ijk;l}(y)u^iv^jw^kz^l,$$

where $u = u^i \frac{\partial}{\partial x^i}|_x$ and etc. We immediately obtain the following

Proposition 13.5.3 *For any locally projectively flat R-flat Finsler metric on an n-dimensional manifold M, the following conditions are equivalent: (a) $\bar{\mathbf{L}} = 0$; (b) $\bar{\mathbf{B}} = 0$; (c) $\mathbf{E} = 0$; (d) $\mathbf{B} = 0$; (e) $\mathbf{L} = 0$; (f) L is locally Minkowskian.*

Proof: Assume that L is a locally projective flat R-flat Finsler metric. (13.63) simplifies to

$$L_{jkl;m} = -\frac{4L}{(n+1)^2}\Big\{E_{jm}E_{kl} + E_{jl}E_{km} + E_{jk}E_{lm}\Big\}. \tag{13.64}$$

Assume that $\bar{\mathbf{L}} = 0$. It follows from (13.64) that

$$E_{jm}E_{kl} + E_{jl}E_{km} + E_{jk}E_{lm} = 0.$$

By the same argument as in the proof of Proposition 13.5.2, we can show that $\bar{\mathbf{B}} = 0$, $\mathbf{E} = 0$, $\mathbf{B} = 0$ and hence $\mathbf{L} = 0$. Thus L is locally Minkowskian by Proposition 8.1.6. Q.E.D.

Now we turn to the main problem in projective geometry of Finsler spaces, that is, to characterize and study locally projectively flat Finsler metrics. This problem remains open, except for Landsberg metrics.

According to Proposition 13.2.2 (and [Bw5] in the two-dimensional case), locally projectively flat Landsberg metrics must be Berwaldian. Thus, there is no difference between "Landsbergian" and "Berwaldian" for locally projectively flat Finsler metrics.

Theorem 13.5.4 *Let L be a locally projectively flat Landsberg metric on an n-dimensional manifold M. Then*

(a) $(n \geq 3)$. L is either Riemannian with constant curvature $\lambda \neq 0$, or locally Minkowskian ;

(b) $(n = 2)$ ([Bw5]). L is Riemannian with constant curvature $\lambda \neq 0$, or locally Minkowskian, or a Finsler metric with $\mathbf{K} \neq$ constant. In the later case, L is in the following form

$$L := \frac{\Big(u + f(x,y)v\Big)^4}{v^2}, \tag{13.65}$$

where $f = f(x,y)$ satisfies the equation $x + yf = \psi(f)$ for some function ψ with $\psi''(z) \neq 0$.

Proof: We will only prove the case when $n \geq 3$. First, we assume that the Ricci scalar $R \neq 0$ at some point. By Proposition 11.1.2, the Finsler metric must be Riemannian, since L is of scalar curvature $\lambda(y) = R(y)/L(y)$. In this case, $\lambda = constant \neq 0$. By continuity, L is Riemannian with constant curvature $\lambda \neq 0$ over the whole manifold. The case when $R = 0$ follows from Proposition 13.5.3.

In Example 13.6.3 below, we will show that the Finsler metric in (13.65) is a locally projectively flat Landsberg metric. See [Bw5] for a complete proof in the two-dimensional case. Q.E.D.

At the end, let us take a look at the following

Example 13.5.1 Let $\mathcal{U}$ be an open subset in R^3 and (x,y,z,u,v,w) the standard coordinate system in $T\mathcal{U} = \mathcal{U} \times \mathrm{R}^3$. Consider the following Finsler metric

$$\begin{aligned} L &= u^2\Big[A\Big(\frac{v}{u}\Big)^2 + 2B\Big(\frac{v}{u}\Big)\Big(\frac{w}{u}\Big) + C\Big(\frac{w}{u}\Big)^2 + \Big(P(x) + Q(y)\Big)\frac{v}{u} \\ &\quad + \Big(U(x) + V(z)\Big)\frac{w}{u} + a(x)y + b(x)z\Big]^2. \end{aligned} \tag{13.66}$$

where A, B and C are constants satisfying $AC - B^2 \neq 0$. According Example 13.2.2, L is a Douglas metric. In fact, $\Phi = \Phi^2(x,y,z,\xi,\zeta)$ and $\Psi = \Phi^3(x,y,z,\xi,\zeta)$ are given by the following simple formulas

$$\begin{aligned} \Phi &= \frac{1}{2B^2}\Big(CP_x - BU_x + Bb - Ca\Big) \\ \Psi &= \frac{1}{2B}\Big(a - P_x\Big). \end{aligned}$$

They are actually independent of ξ and ζ. Plugging them into (13.43) yields

$$\hat{R}^a_b = 0.$$

By Proposition 13.4.3 and Theorem 13.5.1, L is locally projectively flat. One can verify that L is not R-flat, nor Landsbergian. $\sharp$

13.6 Berwald-Weyl Curvature

It follows from Proposition 13.4.1 that the Weyl curvature of any two-dimensional spray always vanishes. Thanks to L. Berwald [Bw8] who introduced a non-trivial projective invariant for a two-dimensional spray $\mathbf{G}$ using the associated local projective spray Π in a local coordinate system (see (12.8)) [Bw8]. As we know, any invariant defined by Π is a projective invariant within the same local coordinate system. However, it might not be globally defined. Berwald did not give the detailed argument for his projective invariant being globally defined.

To avoid this problem, we will use the globally defined associated projective spray $\tilde{\mathbf{G}} = \mathbf{G} + \frac{2\mathbf{S}}{n+1}\mathbf{Y}$ (see (12.10) and (12.12)) to define projective invariants for two-dimensional sprays. Since $\tilde{\mathbf{G}}$ is a modification of $\mathbf{G}$ with the S-curvature $\mathbf{S}$ of $(\mathbf{G}, d\mu)$, where $d\mu$ is a fixed volume form $d\mu$, projective invariants defined by $\tilde{\mathbf{G}}$ possibly depend on $d\mu$. However, this is not a problem here for the projective invariant we are going to introduce.

Let M be an n-dimensional manifold. Fix a volume form $d\mu = \sigma(x)dx^1 \cdots dx^n$ on M. Let $\mathbf{G}$ be a spray on M and

$$\tilde{\mathbf{G}} := \mathbf{G} + \frac{2\mathbf{S}}{n+1}\mathbf{Y}$$

the projective spray associated with $\mathbf{G}$, where $\mathbf{S}$ denotes the S-curvature of $(\mathbf{G}, d\mu)$. Following Berwald [Bw8], we will define a non-trivial invariant using $\tilde{\mathbf{G}}$ instead of Π.

Let $\tilde{R} := \frac{1}{n-1}\widetilde{\mathbf{Ric}}$ which is the Ricci curvature of $\tilde{\mathbf{G}}$ divided by $n-1$. By the definition, we see that $\tilde{R}$ depends on the volume form $d\mu$. Define $\mathbf{W}^o_y : T_xM \to$ R by

$$\mathbf{W}^o_y(u) := W^o_i(y)u^i, \qquad u = u^i \frac{\partial}{\partial x^i}|_x,$$

where

$$W^o_i(y) := \frac{1}{2} y^j \tilde{R}_{\cdot i;j} - \tilde{R}_{;i}. \tag{13.67}$$

The covariant derivatives of $\tilde{R}$ are taken with respect to $\tilde{\mathbf{G}}$, namely,

$$\tilde{R}_{;i} := \tilde{R}_{x^i} - \tilde{N}^j_i \tilde{R}_{y^j}, \qquad \tilde{R}_{\cdot i} := \frac{\partial \tilde{R}}{\partial y^i}$$

$$\tilde{R}_{\cdot i;j} := \frac{\partial \tilde{R}_{\cdot i}}{\partial x^j} - \tilde{N}^k_j \frac{\partial \tilde{R}_{\cdot i}}{\partial y^k} - \tilde{\Gamma}^k_{ij} \tilde{R}_{\cdot k}.$$

We have the following more direct formula for $\mathbf{W}^o$.

$$W^o_i = -\frac{1}{2}\Big\{ 2\tilde{R}_{x^i} - y^j \tilde{R}_{x^j y^i} + 2\tilde{G}^k \tilde{R}_{y^j y^k} - \tilde{N}^k_i \tilde{R}_{y^k} \Big\}. \tag{13.68}$$

Since $\tilde{\mathbf{G}}$ is projectively invariant, $\mathbf{W}^o := \{\mathbf{W}^o_y\}_{y \in TM\setminus\{0\}}$ is a projective invariant. It follows from the homogeneity of $\tilde{R}$ that

$$y^i \tilde{R}_{\cdot i;j} = 2\tilde{R}_{;j}.$$

Thus

$$y^i W^o_i = \frac{1}{2} y^i y^j \tilde{R}_{.i;j} - y^i \tilde{R}_{;i} = y^j \tilde{R}_{;j} - y^i \tilde{R}_{;i} = 0.$$

This gives

$$\mathbf{W}^o_y(y) = 0. \tag{13.69}$$

We call $\mathbf{W}^o$ the *Berwald-Weyl curvature* of $\mathbf{G}$. It turns out that this invariant is independent of the volume form and hence exactly the same invariant as defined by Berwald using Π. That is, Berwald's invariant is indeed a globally defined projective invariant.

Lemma 13.6.1 $\mathbf{W}^o$ *is independent of the volume form* $d\mu$.

We will not give a proof here for all dimensions. We will only calculate $\mathbf{W}^o$ in dimension two to show that it is independent of the volume form.

Let $\mathbf{G}$ be a two-dimensional spray. In a standard local coordinate system (x, y, u, v), express $\mathbf{G}$ as follows

$$\mathbf{G} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - 2G(x,y,u,v)\frac{\partial}{\partial u} - 2H(x,y,u,v)\frac{\partial}{\partial v}.$$

Let

$$\Phi(x,y,\xi) := 2\xi G(x,y,1,\xi) - 2H(x,y,1,\xi). \tag{13.70}$$

We construct a local spray

$$\hat{\mathbf{G}} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - 2\hat{G}(x,y,u,v)\frac{\partial}{\partial u} - 2\hat{H}(x,y,u,v)\frac{\partial}{\partial v}, \tag{13.71}$$

where

$$\hat{G} := 0, \qquad \hat{H} := -\frac{1}{2}u^2\Phi\Big(x, y, \frac{v}{u}\Big).$$

Since $\hat{\mathbf{G}}$ is pointwise projective to $\mathbf{G}$ and $\mathbf{W}^o$ is a projective invariant, we just need to calculate the $\mathbf{W}^o$ for $\hat{\mathbf{G}}$. Fix an area form $d\mu = \sigma(x,y)dxdy$. The S-curvature $\hat{\mathbf{S}}$ of $(\hat{\mathbf{G}}, d\mu)$ is given by

$$\hat{\mathbf{S}} = -\frac{1}{2}u\Phi_\xi - \frac{\sigma_x}{\sigma}u - \frac{\sigma_y}{\sigma}v.$$

Thus

$$\tilde{\mathbf{G}} := \hat{\mathbf{G}} + \frac{2\hat{\mathbf{S}}}{3}\mathbf{Y} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - 2\tilde{G}(x,y,u,v)\frac{\partial}{\partial u} - 2\tilde{H}(x,y,u,v)\frac{\partial}{\partial v},$$

is given by

$$\begin{aligned} u^{-2}\tilde{G} &= \frac{1}{6}\Phi_\xi + \frac{\sigma_x + \sigma_y\xi}{3\sigma} \\ u^{-2}\tilde{H} &= -\frac{1}{2}\Phi + \frac{1}{6}\xi\Phi_\xi + \frac{\sigma_x + \sigma_y\xi}{3\sigma}\xi, \end{aligned}$$

where $\xi = v/u$. The Ricci scalar $\tilde{R} = \widetilde{\mathbf{Ric}}$ of $\tilde{\mathbf{G}}$ is given by

$$\tilde{R} = 2(\tilde{G}_x+\tilde{H}_y)-P_xu-P_yv+2P_u\tilde{G}+2P_v\tilde{H}-(\tilde{G}_u\tilde{G}_u+2\tilde{G}_v\tilde{H}_u+\tilde{H}_v\tilde{H}_v), \quad (13.72)$$

where $P := \tilde{G}_u + \tilde{H}_v$. A direct computation gives

$$\begin{aligned} u^{-2}\tilde{R} \;=\; & -\Phi_y + \frac{1}{3}\Phi_{x\xi} + \frac{1}{3}\xi\Phi_{y\xi} + \frac{1}{3}\Phi\Phi_{\xi\xi} - \frac{2}{9}\Phi_\xi\Phi_\xi \\ & -\frac{1}{3}\frac{\sigma_y}{\sigma}\Phi + \frac{1}{9}\Big[\frac{\sigma_x+\sigma_y\xi}{\sigma}\Big]\Phi_\xi \\ & -\frac{\sigma_{xx}+2\sigma_{xy}\xi+\sigma_{yy}\xi^2}{2\sigma} + \frac{4}{9}\Big[\frac{\sigma_x+\sigma_y\xi}{\sigma}\Big]^2. \end{aligned} \quad (13.73)$$

Thus $\tilde{R}$ still depends on the volume form $d\mu$.

Now we compute the Berwald-Weyl curvature. It follows from (13.69) that

$$W_1^o = -W_2^o\xi.$$

Thus we just need to compute W_2^o. By definition,

$$W_2^o = -\frac{1}{2}\Big\{2\tilde{R}_y - u\tilde{R}_{xv} - v\tilde{R}_{yv} + 2\tilde{G}\tilde{R}_{uv} + 2\tilde{H}\tilde{R}_{vv} - \tilde{G}_v\tilde{R}_u - \tilde{H}_v\tilde{R}_v\Big\}. \quad (13.74)$$

Using (13.73), we immediately obtain

$$\begin{aligned} u^{-2}W_2^o \;=\; & \frac{1}{6}\Phi_{xx\xi\xi} + \frac{1}{3}\xi\Phi_{xy\xi\xi} + \frac{1}{6}\Phi_x\Phi_{\xi\xi\xi} + \frac{1}{3}\Phi\Phi_{x\xi\xi\xi} \\ & +\frac{1}{6}\xi^2\Phi_{yy\xi\xi} + \frac{1}{6}\xi\Phi_y\Phi_{\xi\xi\xi} + \frac{1}{3}\xi\Phi\Phi_{y\xi\xi\xi} + \frac{1}{6}\Phi^2\Phi_{\xi\xi\xi\xi} \\ & -\frac{1}{2}\Phi\Phi_{y\xi\xi} - \frac{1}{6}\Phi_\xi\Phi_{x\xi\xi} - \frac{1}{6}\xi\Phi_\xi\Phi_{y\xi\xi} + \frac{2}{3}\Phi_\xi\Phi_{y\xi} \\ & -\frac{2}{3}\xi\Phi_{yy\xi} - \frac{2}{3}\Phi_{xy\xi} - \frac{1}{2}\Phi_y\Phi_{\xi\xi} + \Phi_{yy}. \end{aligned} \quad (13.75)$$

Yes, W_2^o is independent of $d\mu$! This is exactly the invariant that L. Berwald defined using the local projective spray Π in a local coordinate system. Q.E.D.

In [Bw5], L. Berwald introduced another non-trivial projective invariant for two-dimensional Finsler metrics, which is in fact also defined for sprays.

Let $\mathbf{G}$ be a spray on an n-dimensional manifold M. Let

$$R_{\cdot i} := \frac{\partial R}{\partial y^i}, \qquad \tilde{R}_i := \frac{1}{3}\Big\{(n-1)R_{\cdot i} - \frac{\partial R^m_i}{\partial y^m}\Big\} \quad (13.76)$$

where $R = \frac{1}{n-1}\tilde{R}_i y^i = \frac{1}{n-1}R^m_m$ is the Ricci scalar. For a vector $y \in T_xM$, define a skew-symmetric bilinear form $\widetilde{\mathbf{W}}_y : T_xM \times T_xM \to \mathrm{R}$ by

$$\widetilde{\mathbf{W}}_y(u,v) := \widetilde{W}_{ij}(y)u^iv^j \qquad u = u^i\frac{\partial}{\partial x^i}|_x,\; v = v^j\frac{\partial}{\partial x^j}|_x, \quad (13.77)$$

where

$$\widetilde{W}_{ij} := \frac{1}{3}\Big\{ \Big(\tilde{R}_{i;j} - \tilde{R}_{j;i}\Big) + (n-1)\Big(R_{\cdot i;j} - R_{\cdot j;i}\Big)\Big\}. \tag{13.78}$$

Berwald proved that $\widetilde{\mathbf{W}} = \{\widetilde{\mathbf{W}}_y\}_{y\in TM\setminus\{0\}}$ is a projective invariant. As a matter of fact, in dimension two,

$$\widetilde{W}_{21} = W^o_2. \tag{13.79}$$

Thus for two-dimensional sprays

$$\widetilde{\mathbf{W}} = 0 \qquad \Longleftrightarrow \qquad \mathbf{W}^o = 0.$$

Therefore we obtain the following

L. Berwald proved the following remarkable result.

Theorem 13.6.2 ([Bw5][Bw7][Bw8]) *For a two-dimensional spray* $\mathbf{G}$*, the following are equivalent*

(a) $\mathbf{G}$ *is locally projectively flat;*

(b) $\mathbf{D} = 0$ *and* $\mathbf{W}^o = 0$*;*

(c) $\mathbf{D} = 0$ *and* $\widetilde{\mathbf{W}} = 0$.

By (13.78), we immediately obtain the following

Corollary 13.6.3 *For any two-dimensional spray with* $\mathbf{R} = 0$ *or any two-dimensional Finsler metric of constant curvature,*

$$\mathbf{W}^o = 0 = \widetilde{\mathbf{W}}.$$

The converse of Corollary 13.6.3 is not true. There exist two-dimensional projectively flat Finsler metrics whose Gauss curvature is not constant, even the main scalar is not constant. See (13.88) and (13.92) below.

Remark 13.6.4 Consider a spray on an open subset $\mathcal{U} \subset \mathrm{R}^2$

$$\mathbf{G} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - 2G(x,y,u,v)\frac{\partial}{\partial u} - 2H(x,y,u,v)\frac{\partial}{\partial v}.$$

Suppose that the function Φ defined in (13.70) satisfies

$$\Phi_y = 0, \qquad \Phi_{x\xi\xi} + \Phi\Phi_{\xi\xi\xi} = 0, \tag{13.80}$$

First, by $\Phi_y = 0$, we simplify (13.75) to

$$u^{-1}W^o_2 = \frac{1}{6}\Psi_x + \frac{1}{6}\Phi\Psi_\xi - \frac{1}{6}\Phi_\xi\Psi,$$

where $\Psi := \Phi_{x\xi\xi} + \Phi\Phi_{\xi\xi\xi}$. Now by $\Psi = 0$, we obtain that $\mathbf{W}^o = 0$. In addition, if Φ is a polynomial of degree three or less in ξ, then $\mathbf{G}$ is locally projectively flat.

Suppose that $\mathbf{G}$ is a two-dimensional Douglas spray. Then Φ is in the following form

$$\Phi = A(x,y) + B(x,y)\xi + C(x,y)\xi^2 + D(x,y)\xi^3, \tag{13.81}$$

In this case, W_2^o is given by

$$\begin{aligned} u^{-2}W_2^o \;=\; & DA_x + 2D_xA + \frac{1}{3}C_{xx} - \frac{1}{3}C_xB - CA_y - C_yA \\ & -\frac{2}{3}B_{xy} + \frac{2}{3}BB_y + A_{yy} + \Big(D_{xx} + D_xB + DB_x - 2DA_y \\ & -D_yA - \frac{2}{3}C_{xy} - \frac{2}{3}CC_x + \frac{1}{3}CB_y + \frac{1}{3}B_{yy}\Big)\xi. \end{aligned} \tag{13.82}$$

By Theorem 13.6.2, we obtain the following

Corollary 13.6.5 ([Bw8]) *Let* $\mathbf{G}$ *be a two-dimensional Douglas spray with* Φ *in the form of (13.81).* $\mathbf{G}$ *is locally projectively flat if and only if* A,B *and* C *satisfy the following equations*

$$DA_x + 2D_xA + \frac{1}{3}C_{xx} - \frac{1}{3}C_xB - CA_y - C_yA - \frac{2}{3}B_{xy} + \frac{2}{3}BB_y + A_{yy} = 0$$
$$D_{xx} + D_xB + DB_x - 2DA_y - D_yA - \frac{2}{3}C_{xy} - \frac{2}{3}CC_x + \frac{1}{3}CB_y + \frac{1}{3}B_{yy} = 0.$$

A two-dimensional spray $\mathbf{G}$ on a surface M is called a *constant affine spray* if at every point, there is a local coordinate system (x,y) in M such that in the standard local coordinate system (x,y,u,v) in TM

$$\mathbf{G} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - 2G(x,y,u,v)\frac{\partial}{\partial u} - 2H(x,y,u,v)\frac{\partial}{\partial v}$$

is in the form

$$G = \Gamma^1_{11}u^2 + 2\Gamma^1_{12}uv + \Gamma^1_{22}v^2, \qquad H = \Gamma^2_{11}u^2 + 2\Gamma^2_{12}uv + \Gamma^2_{22}v^2$$

where Γ^i_{jk} are constants. Let Φ be defined in (13.70). It is easy to see that Φ must be in the form (13.81) with constant coefficients A, B, C and D. Thus $\mathbf{W}^o = 0$ by (13.75) or (13.82). We obtain the following

Corollary 13.6.6 ([AnHaMo]) *Every two-dimensional constant spray is locally projectively flat.*

According to [AnHaMo], Corollary 13.6.6 is not true for higher dimensional constant affine sprays.

Remark 13.6.7 Consider a Finsler metric $L = L(x, y, u, v)$ on an open subset $\mathcal{U} = (a, b) \times (c, d) \subset \mathrm{R}^2$ in the following form

$$L = \Big[u\phi\Big(x, y, \frac{v}{u}\Big)\Big]^2.$$

$\phi = \phi(x, y, \xi)$ must be a Lagrange metric on $\mathcal{U}$. Let

$$\Phi := \frac{\phi_y - \phi_{x\xi} - \phi_{y\xi}\xi}{\phi_{\xi\xi}}. \tag{13.83}$$

By Proposition 3.3.1, the spray coefficients $G = G(x, y, u, v)$ and $H = H(x, y, u, v)$ of L is related to $\Phi = \Phi(x, y, \xi)$ by (8.83), i.e.,

$$\Phi(x, y, \xi) = 2\xi G(x, y, 1, \xi) - 2H(x, y, 1, \xi). \tag{13.84}$$

By (13.75), the Berwald-Weyl curvature $\mathbf{W}^o$ of L can be expressed in terms of ϕ and its partial derivatives directly. However, it contains too many terms, so that we are not going to give all the terms here. Assume that ϕ satisfies (8.83),

$$\phi_y = 0, \qquad \frac{\phi_x}{\phi^2} + \Big(\frac{\phi_{x\xi}}{\phi\phi_{\xi\xi}}\Big)_\xi = 0. \tag{13.85}$$

Then Φ satisfies (13.80). Thus $\mathbf{W}^o = 0$. By Proposition 8.2.7, (13.85) also implies that the Riemann curvature of L vanishes, $\mathbf{R} = 0$. In this case, L is locally projectively flat if and only if $\Phi = -\phi_{x\xi}/\phi_{\xi\xi}$ is a polynomial of degree three or less in ξ.

A Finsler metric is called a *constant Berwald metric* if the induced spray is a constant affine spray. By Corollary 13.6.6, any two-dimensional constant-Berwald metric must be locally projectively flat. Antonelli-Matsumoto [AnMa2] have proved that any two-dimensional constant-Berwald metric is either locally Minkowskian or pseudo-Riemannian.

Example 13.6.1 Consider the following Kropina metric on an open subset $\mathcal{U} = (a, b) \times (c, d) \subset \mathrm{R}^2$

$$L := \Big[a(x, y)\frac{v^2}{u} + b(x, y)v + c(x, y)u\Big]^2.$$

The corresponding function $\Phi = \Phi(x, y, \xi)$ in (13.83) is given by

$$\Phi = -\frac{1}{2a}\Big\{a_y\xi^2 + 2a_x\xi + b_x - c_y\Big\}. \tag{13.86}$$

Thus L is a Douglas metric. One can check that L is not a Berwald metric. By Theorem 13.6.2, L is locally projectively flat if the function Φ in (13.86) satisfies

$$W_2^o = 0.$$

We compute W_2^o for Φ as follows

$$\begin{aligned} u^{-2}W_2^o &= \frac{1}{4a^3}\Big\{2a_xa_xa_y - 2a_ya_yb_x + 2a_ya_yc_y \\ &-2a\Big(a_xa_{xy} + a_ya_{xx}\Big) + 2a^2\Big(a_{xxy} - b_{xyy} + c_{yyy}\Big) \\ &+a\Big(a_{yy}b_x - a_{yy}c_y + 3a_yb_{xy} - 3a_yc_{yy}\Big)\Big\} \end{aligned}$$

Assume that $a = a(x)$ is a function of x only. Then W_2^o simplifies to

$$W_2^o = -\frac{1}{2a}\Big(b_{xyy} - c_{yyy}\Big)u^2.$$

Thus the following Kropina metric

$$L := \Big[a(x)\frac{v^2}{u} + b(x,y)v + c(x,y)u\Big]^2 \tag{13.87}$$

is locally projectively flat if and only if

$$b_{xyy} = c_{yyy}.$$

In particular, the Kropina metric in (3.47) is locally projectively flat. See [Ma11][BaMa2] for related discussion.

Consider a special form of (13.87)

$$L := e^{\rho(x)}\Big[\frac{v^2}{u} + bv + cu\Big]^2, \tag{13.88}$$

where b and c are constants. By the above argument, we know that L is projectively flat. Note that it is also conformally flat, namely, it is conformal to a Minkowski functional. However, the Gauss curvature is not constant. Moreover, the main scalar of L at each point is not a constant.

Let $m \geq 2$ be an integer. Consider the following Finsler metric

$$L := \Big[a(x)\frac{v^m}{u^{m-1}} + e_x(x,y)u + e_y(x,y)v\Big]^2, \tag{13.89}$$

where $e = e(x,y)$ is an arbitrary function. A direct computation yields

$$\Phi = -\frac{1}{m^2 - m}\frac{a_x}{a}\xi. \tag{13.90}$$

Thus L is always a Douglas metric. Plugging Φ into (13.75) gives $W_2^o = 0$. Thus L is locally projectively flat. ♯

Example 13.6.2 Let $F = F(x, y, u, v)$ be a Randers metric on an open subset $\mathcal{U} = \{(x, y),\ y > x^2\}$ in the form

$$F = \frac{\sqrt{(2xu - v)^2 + 4(y - x^2)u^2} - (v - 2xu)}{2(y - x^2)}. \tag{13.91}$$

The spray coefficients $G = G^1$ and $H = G^2$ are quite complicated, but in a special form

$$G = P(x, y, u, v)u, \qquad H = P(x, y, u, v)v,$$

where

$$P = \frac{\sqrt{(v - 2xu)^2 + 4(y - x^2)u^2}}{2(y - x^2)} - \frac{u^2}{\sqrt{(v - 2xu)^2 + 4(y - x^2)u^2} + (2xu - v)}$$

Thus the function defined by either (13.83) or (13.84) must vanish, $\Phi = 0$. This implies that the geodesics of F are straight lines, and F is pointwise projectively flat. Further computation shows that the Gauss curvature $\mathbf{K} = -\frac{1}{4}$. But F is not a Berwald metric. The main scalar is not constant either. ♯

Example 13.6.3 ([Bw5]) Let $L = L(x, y, u, v)$ be a Finsler metric on an open subset $\mathcal{U} \subset \mathrm{R}^2$ in the form

$$L := \frac{\Big(u + f(x, y)v\Big)^4}{v^2}, \tag{13.92}$$

The spray coefficients $G = G^1$ and $H = G^2$ of L are given by

$$\begin{aligned} G &= f_x uv + \frac{1}{2}(f_y - f f_x)v^2 \\ H &= f_x v^2. \end{aligned}$$

Thus L is a Berwald metric. The corresponding $\Phi = \Phi(x, y, \xi)$ defined by either (13.83) or (13.84) is given by

$$\Phi = (f_y - f f_x)\xi^3.$$

Plugging it into (13.75) gives

$$W_2^o = \Big(f_{xxy} - 3f_x f_{xx} - f f_{xxx}\Big)uv.$$

Thus L is locally projectively flat if and only if

$$f_{xxy} - 3f_x f_{xx} - f f_{xxx} = 0. \tag{13.93}$$

One can verify that the main scalar $\mathbf{I}^2 = 9/2$, hence L is Landsbergian ($\mathbf{J} = 0$). But the Gauss curvature is not constant. Note that if f is the function in (13.65), i.e., f satisfies $x + yf = \psi(f)$ for some function ψ with $\psi''(f) \neq 0$, then f satisfies (13.93). This shows that the Finsler metric L in (13.65) is locally projectively flat with $\mathbf{K} \neq$ *constant*. ♯

Finally, let us mention some remarkable results by L. Berwald [Bw5]. He determines all two-dimensional locally projective flat Finsler metrics the main scalar of which is a function of position only. First he establishes some theorems showing that for such a metric the main scalar $\mathbf{I}$ is always a constant and the curvature nearly always a constant. For $\mathbf{I}^2 \neq 0, 9/2$, the Finsler metric is locally Minkowskian. For $\mathbf{I} = 0$, the Gauss curvature $\mathbf{K} = constant$. In this case, the Finsler metric is Riemannian of constant curvature. For $\mathbf{I}^2 = 9/2$, the Gauss curvature can be variable. The Finsler metric is given in (13.92) , where $f = f(x, y)$ satisfies (13.93). It is easy to see that this metric has constant main scalar $\mathbf{I}^2 = 9/2$, but it does not have constant Gauss curvature. Thus there exists 2-dimensional projectively flat Finsler metrics which are not of constant curvature.

Chapter 14

Exponential Maps

In this chapter, we will introduce the notion of exponential maps for spray spaces. The exponential map of a spray space is a map from the tangent bundle of the space into the space. It is defined via geodesics.

14.1 Exponential Map of Sprays

Let $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}$ be a spray on a manifold M. For a vector $y \in TM$, let c_y denote the geodesic of $\mathbf{G}$ with $\dot{c}_y(0) = y$. By definition, c_y is the projection of an integral curve $\hat{c}_y$ of $\mathbf{G}$ in TM with $\hat{c}_y(0) = y$.

$$\frac{d\hat{c}_y}{dt} = \mathbf{G}_{\hat{c}_y}.$$

The local coordinates $(x^i(t), y^i(t))$ of $\hat{c}_y(t)$ satisfy

$$\begin{cases} \frac{dx^i}{dt}(t) & = y^i(t) \\ \frac{dy^i}{dt}(t) & = -2G^i(x(t), y(t)). \end{cases} \tag{14.1}$$

By the O.D.E. theory, $\hat{c}_y$ smoothly depends on $y \in TM \setminus \{0\}$. For any pre-compact domain $K \subset M \subset TM$, there is a number $\varepsilon > 0$ and an open neighborhood $\mathcal{U}$ of K in TM such that $c_y(t)$ is defined on $[0, 1+\varepsilon)$ for any $y \in \mathcal{U} \setminus \{0\}$. Further, $c_y(t)$ is C^∞ on $[0, 1+\varepsilon) \times (\mathcal{U} \setminus \{0\})$. Define a map $\exp : \mathcal{U} \to M$ by

$$\exp(y) := c_y(1).$$

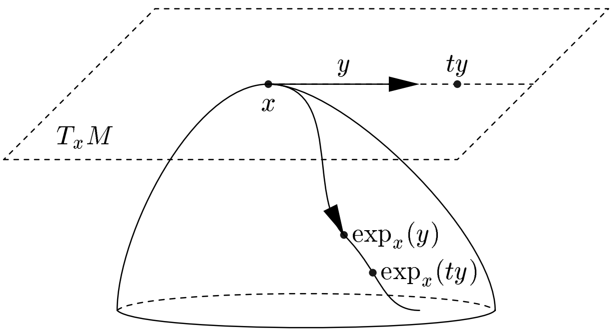
By the homogeneity of $\mathbf{G}$, we can easily show that

$$\exp(ty) = c_{ty}(1) = c_y(t), \qquad t \geq 0.$$

exp is called the *exponential map*. The above argument shows that exp is C^∞ on $\mathcal{U} \setminus \{0\}$. For a point $x \in M$, let $\mathcal{U}_x = \mathcal{U} \cap T_xM$ and

$$\exp_x := \exp|_{\mathcal{U}_x}.$$

We see that $\exp_x$ is C^∞ on $\mathcal{U}_x \setminus \{0\}$.



The differential of $\exp_x$ at a point $y \in \mathcal{U}_x$ is a linear map

$$d(\exp_x)_{|y} : T_y\mathcal{U}_x = T_xM \to T_zM, \qquad z = \exp_x(y).$$

In particular, if $\exp_x$ is differentiable at the origin of T_xM, then

$$d(\exp)_{|0} : T_xM \to T_xM.$$

Theorem 14.1.1 (Whitehead [Wh]) *Let $(M, \mathbf{G})$ be a spray space. The exponential map* exp *is C^1 at any zero section of TM. Further, for any $x \in M$, $d(\exp_x)_{|0}$ is the identity map at the origin $0 \in T_xM$.*

Proof: We shall only sketch the proof. Let

$$\phi(t, y) := \exp(ty), \qquad t \geq 0,\ y \in TM.$$

Note that

$$\phi(\lambda t, y) = \phi(t, \lambda y), \qquad \forall \lambda > 0. \tag{14.2}$$

For simplicity, we set

$$f^k\Big((x^i), (y^i)\Big) := \phi^k(1, y), \qquad y = y^i\frac{\partial}{\partial x^i}|_x.$$

(14.2) gives rise to

$$f^k\Big((x^i), \lambda(y^i)\Big) = \phi^k(\lambda, y). \tag{14.3}$$

By the mean value theorem, there is $0 < t_\lambda < \lambda$ such that

$$\frac{f^k\Big((x^i), \lambda(y^i)\Big) - f^k\Big((x^i), 0\Big)}{\lambda} = \frac{\partial \phi^k}{\partial \lambda}(t_\lambda, y) \to y^k,$$

as $\lambda \to 0^+$. Thus

$$\frac{\partial f^k}{\partial y^j}(x^i, 0) y^j = y^k.$$

This implies that $d(\exp_x)|_0 = identity$ at the origin $0 \in T_xM$.

Differentiating (14.3) with respect to y^j yields

$$\frac{\partial f^k}{\partial y^j}\Big((x^i), \lambda(y^i)\Big) = \frac{1}{\lambda}\frac{\partial \phi^k}{\partial y^j}(\lambda, y). \tag{14.4}$$

Note that

$$\frac{\partial \phi^k}{\partial y^j}(0, y) = 0.$$

We obtain

$$\begin{aligned}\frac{\partial f^k}{\partial y^j}\Big((x^i), \lambda(y^i)\Big) &= \frac{1}{\lambda}\Big[\frac{\partial \phi^k}{\partial y^j}(\lambda, y) - \frac{\partial \phi^k}{\partial y^j}(0, y)\Big] \\ &= \frac{\partial^2 \phi^k}{\partial \lambda \partial y^j}(t_\lambda, y).\end{aligned}$$

Thus

$$\lim_{\lambda \to 0^+} \frac{\partial f^k}{\partial y^j}\Big((x^i), \lambda(y^i)\Big) = \frac{\partial}{\partial y^j}\Big(\frac{\partial \phi^k}{\partial \lambda}(0, y)\Big) = \frac{\partial}{\partial y^j}(y^k) = \delta^k_j.$$

That is, f is C^1 along the set of zero sections of TM. Q.E.D.

By Theorem 14.1.1, we immediately obtain the following

Corollary 14.1.2 *Let* $(M, \mathbf{G})$ *be a spray space. For any point* $x \in M$, *there is a neighborhood* $\mathcal{U}_x$ *of* x *and a positive number* $r_x > 0$ *such that for any* $x \in \mathcal{U}_x$, $\exp_z : T_zM \to M$ *is a diffeomorphism on* $\mathrm{B}_z(r) \subset T_zM$ *onto its image for all* $0 < r < r_x$, *where*

$$\mathrm{B}_z(r) := \Big\{ y^i \frac{\partial}{\partial x^i}\Big|_z \in T_zM, \ \sum_{i=1}^n (y^i)^2 < r^2 \Big\}.$$

Now we study the regularity of exp on zero sections in TM. First, we assume that $\mathbf{G}$ is an affine spray, i.e., $G^i(y)$ are quadratic functions in $y \in T_xM$ for all $x \in M$. By the O.D.E. theory, the solution $\hat{c}_y$ to (14.1) smoothly depends on the initial data. Thus $\exp(y) = c_y(1)$ is C^∞ on $\mathcal{U}$. In particular, $\exp_x$ is C^2 at $0 \in T_xM$ for all $x \in M$.

Conversely, we assume that $\exp_x$ is C^2 at $0 \in T_xM$ for all x. By definition, for any $r > 0$ and compact subset $\bar{\mathcal{U}} \subset M$, there is a number $\varepsilon > 0$ such that $c_y(t) = \exp(ty)$, $0 \le t \le 1 + \varepsilon$, is a geodesic for every $y \in T\bar{\mathcal{U}}$ with $F(y) < r$. Substituting c_y into the geodesic equation (4.6) and letting $t = 0$ yields

$$y^j y^k \frac{\partial^2 (\exp)^i}{\partial y^j \partial y^k}(0) + 2G^i(y) = 0.$$

In other words, $G^i(y)$ are quadratic in $y \in T_xM$. Thus $\mathbf{G}$ is affine. We have proved the following

Theorem 14.1.3 (Akbar-Zadeh [AZ2]) *Let $(M, \mathbf{G})$ be a spray space.* exp *is C^2 at zero sections in TM if and only if $\mathbf{G}$ is affine.*

Recall that a spray $\mathbf{G}$ is said to be positively complete if every geodesic defined on $[0, a)$ can be extended to a geodesic defined on $[0, \infty)$. By the above arguments, we see that if $\mathbf{G}$ is positively complete, then exp is defined on the whole TM. Further, exp is C^∞ on $TM \setminus \{0\}$ and C^1 on the zero sections of TM by Theorem 14.1.1. In particular, $\exp_x : T_xM \to M$ is C^∞ on $T_xM \setminus \{0\}$ and C^1 at the origin $0 \in T_xM$. The question is whether or not $\exp_x$ covers the whole manifold M?

Theorem 14.1.4 (Hopf-Rinow) *Let (M, F) be a positive definite Finsler space. The following are equivalent:*

(a) F is positively complete;

(b) $\exp_x : T_xM \to M$ is an onto map for some $x \in M$;

(c) $\exp_x : T_xM \to M$ is an onto map for any $x \in M$;

(d) every two points $x_1, x_2 \in M$ can be joined by a minimizing geodesic.

The proof is omitted. See [BaChSh1]. The Hopf-Rinow Theorem is, however, not true for positively complete spray spaces. The following example shows that there exist positively complete sprays for which the exponential map $\exp_x$: $T_xM \to M$ is not onto.

Example 14.1.1 Consider the following spray on R^2

$$\mathbf{G} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - v\sqrt{u^2+v^2}\frac{\partial}{\partial u} + u\sqrt{u^2+v^2}\frac{\partial}{\partial v}. \tag{14.5}$$

By Example 4.1.3, the geodesics of $\mathbf{G}$ are circles of radius $r = 1$ on R^2.

$$\begin{aligned} x(t) &= \cos(\lambda t + \theta) - \cos\theta + x(0), \\ y(t) &= \sin(\lambda t + \theta) - \sin\theta + y(0). \end{aligned}$$

Thus $\mathbf{G}$ is a complete spray. Express $\exp_0 : T_0\mathrm{R}^2 = \mathrm{R}^2 \to \mathrm{R}^2$ by

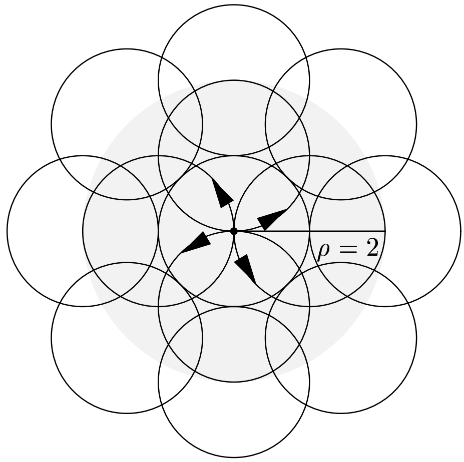
$$\exp_0(u, v) = (\Phi(u, v), \Psi(u, v)).$$

We obtain

$$\Phi(u,v) = \frac{1}{\sqrt{u^2+v^2}}\Big(v\cos\sqrt{u^2+v^2} + u\sin\sqrt{u^2+v^2} - v\Big), \tag{14.6}$$

$$\Psi(u,v) = \frac{1}{\sqrt{u^2+v^2}}\Big(-u\cos\sqrt{u^2+v^2} + v\sin\sqrt{u^2+v^2} + u\Big). \tag{14.7}$$

Note that geodesics issuing from the origin form a disk of radius $\rho = 2$. This disk is the image of $T_0\mathrm{R}^2 = \mathrm{R}^2$ under $\exp_0$. In other words, $\exp_0$ does not cover R^2, although $\mathbf{G}$ is defined on the whole R^2.



Suppose that the spray $\mathbf{G}$ in (14.5) is globally pointwise projective to a positive definite Finsler metric F on R^2. Since the geodesics of $\mathbf{G}$ and F are circles, F is complete. By the Hopf-Rinow Theorem, the exponential map of F (which is also the exponential map of $\mathbf{G}$) must cover the whole manifold. This is impossible. Thus $\mathbf{G}$ can not be globally pointwise projective to any positive definite Finsler metric on R^2. Since $\mathbf{G}$ is isotropic, according to [GrMu], it is locally Finslerian, i.e., it can be locally induced by a (possibly not positive definite) Finsler metric. ♯

14.2 Jacobi Fields in Spray Spaces

In this section, we will discuss some basic properties of Jacobi fields induced by the exponential map.

Let $\mathbf{G}$ be a positively complete spray on an n-manifold M. At every point $x \in M$,

$$\exp_x : T_xM \to M$$

is defined on the whole T_xM. Take an arbitrary vector $y \in T_xM \setminus \{0\}$. The curve $c(t) := \exp_x(ty)$ is a geodesic issuing from x. For a vector $v \in T_xM$, consider the following special geodesic variation of c

$$H(s,t) := \exp_x\big(t(y+sv)\big).$$

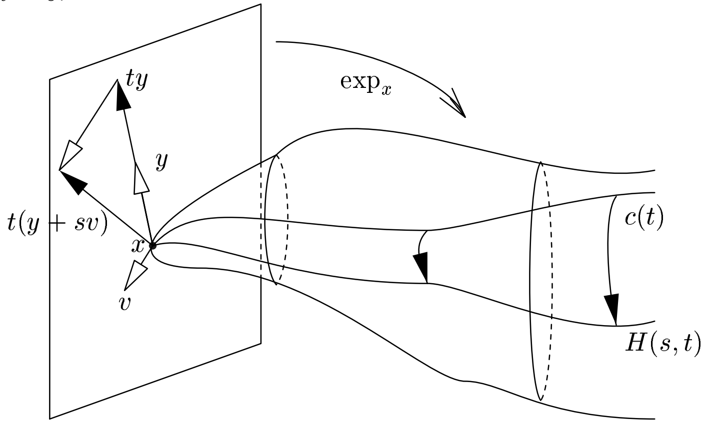
By Lemma 8.1.1, we know that

$$J(t) := d(\exp_x)_{|ty}(tv) \tag{14.8}$$

is a Jacobi field along $c(t)$, $0 < t < \infty$. Namely, it satisfies

$$\mathrm{D}_{\dot c}\mathrm{D}_{\dot c}J + \mathbf{R}_{\dot c}(J) = 0. \tag{14.9}$$

Since $\exp_x$ is C^∞ on $T_xM \setminus \{0\}$ and C^1 at the origin, J is C^∞ on $(0, \infty)$ and C^0 at $t = 0$.



Lemma 14.2.1 *Let $c : [-\delta, \infty) \to M$ be a geodesic with $\dot c(0) = y \in T_xM$. For any $v \in T_xM$, where $\delta > 0$, the Jacobi field $J(t) := d(\exp_x)_{|ty}(tv)$, $0 < t \le \infty$, can be extended smoothly across $t = 0$ to a Jacobi field $\tilde J$ on $[-\delta, \infty)$ such that*

$$J(0) = 0, \qquad \mathrm{D}_y J(0) = v. \tag{14.10}$$

Proof: By assumption, J satisfies (14.9). Fix a number $t_0 \in (0, \infty)$. Since $\mathbf{R}_{\dot c}$ is C^∞ on $[-\delta, \infty)$, there exists an unique solution $\tilde J$ of (14.9) on $[-\delta, \infty)$ satisfying the initial data:

$$\tilde J(t_o) = J(t_o), \qquad D_{\dot c}\tilde J(t_o) = D_{\dot c}J(t_o).$$

By the Uniqueness Theorem in O.D.E. theory, we have

$$\tilde J(t) = J(t), \qquad t \in (0, \infty).$$

In other words, J can be extended to a Jacobi field $\tilde J$ smoothly across $t = 0$.

Since $J(0) = 0$, we can write

$$\tilde J(t) = tW(t), \qquad -\delta \le t \le \infty, \tag{14.11}$$

where $W(t)$ is a C^∞ vector field along c. Observe that for $0 < t < \infty$,

$$tW(t) = d(\exp_x)_{|ty}(tv).$$

Thus

$$W(t) = d(\exp_x)_{|ty}(v), \quad 0 < t < \infty. \tag{14.12}$$

By Theorem 14.1.1, we know that $d(\exp_x)_{|0} = identity$. Thus

$$W(0) = \lim_{t\to 0^+} d(\exp_x)_{|ty}(v) = v$$

Differentiating (14.11) yields

$$\mathrm{D}_{\dot c} J(t) = W(t) + t\mathrm{D}_{\dot c} W(t).$$

Thus

$$\mathrm{D}_y J(0) = W(0) = v.$$

Q.E.D.

By Lemma 14.2.1, we obtain the following

Lemma 14.2.2 *Let $c(t) := \exp_x(ty)$, $0 \le t < \infty$. For a number $r > 0$, $\exp_x$ is singular at ry if and only if there exists a Jacobi field $J \not\equiv 0$ along c, $0 \le t \le r$, such that*

$$J(0) = 0 = J(r). \tag{14.13}$$

Definition 14.2.3 Let $(M, \mathbf{G})$ be a positively complete spray space. A vector $y \in T_xM \setminus \{0\}$ is called a *conjugate vector* of x if

$$d(\exp_x)_{|y} : T_y(T_xM) \to T_{\exp_x(y)}M$$

is singular, and $d(\exp_x)_{|ty}$ is regular for $0 < t < 1$. Denote by $\mathrm{Conj}(x)$ the set of all conjugate vectors at x. When $\mathbf{G}$ is induced by a Finsler metric L, for a vector $y \in T_xM$ with $L(y) = \pm 1$, denote by $\mathbf{c}_y > 0$ the number such that $\mathbf{c}_y y$ is a conjugate vector. We call $\mathbf{c}_y$ the *conjugate value* of y.

Given a spray space $(M, \mathbf{G})$, we may ask the following question: Under what condition on the geometric quantities of $\mathbf{G}$, $\exp_x$ is non-singular on T_xM?

Let $c(t) = \exp_x(ty)$, $0 \le t < \infty$, be a geodesic issuing from x and $J(t)$ be a Jacobi field along c. Take a parallel frame $\{E_i(t)\}_{i=1}^n$ along c, namely, $\mathrm{D}_{\dot c} E_i(t) = 0$ and $\{E_i(t)\}_{i=1}^n$ is a basis for $T_{c(t)}M$ for all t. Write

$$J(t) = J^i(t)E_i(t)$$

and

$$\mathbf{R}_{\dot c(t)}(E_k(t)) = R^i_k(t)E_i(t).$$

Equation (14.9) becomes

$$\frac{d^2 J^i}{dt^2}(t) + R^i_k(t)J^k(t) = 0. \tag{14.14}$$

If all solutions $(J^i(t))$ to (14.14) with $(J^i(0)) = 0$ and $(\dot J^i(0)) \ne 0$ do not vanish at any $t > 0$, then $\exp_x$ is non-singular along the ray ty, $t > 0$.

14.3 Comparison Theorems for Finsler Spaces

In this section, we will discuss geometric meaning of the sign of the Riemann curvature. Only positive definite Finsler metrics will be considered. We denote a positive definite Finsler metric L on a manifold M by

$$F(y) := \sqrt{L(y)}, \qquad y \in TM.$$

F has non-positive curvature $\mathbf{K} \le 0$ if for any flag $(P, y) \subset T_xM$, the flag curvature is nonpositive

$$\mathbf{K}(P, y) \le 0.$$

Using Proposition 8.2.1, we obtain the following important results for Jacobi fields along a geodesic.

Lemma 14.3.1 (Gauss Lemma) *Let (M, F) be a Finsler space and $c : (a, b) \to M$ a geodesic. Let J be a Jacobi field along c. Let $y = \dot{c}(t_o)$, $u = J(t_o)$ and $v = \mathrm{D}_y J(t_o)$, where $t_o \in (a, b)$. Then*

$$g_{\dot{c}(t)}(J(t), \dot{c}(t)) = g_y(v, y)(t - t_o) + g_y(u, y). \tag{14.15}$$

Proof: Since c is geodesic, $\mathrm{D}_{\dot{c}}\dot{c} = 0$. By (7.11), (8.2) and (8.57), we have

$$\begin{aligned} \frac{d}{dt}\Big(g_{\dot{c}}(J, \dot{c})\Big) &= g_{\dot{c}}\Big(\mathrm{D}_{\dot{c}}J, \dot{c}\Big), \\ \frac{d^2}{dt^2}\Big(g_{\dot{c}}(J, \dot{c})\Big) &= g_{\dot{c}}(\mathrm{D}_{\dot{c}}\mathrm{D}_{\dot{c}}J, \dot{c}) \\ &= -g_{\dot{c}}(\mathbf{R}_{\dot{c}}(J), \dot{c}) \\ &= -g_{\dot{c}}(\mathbf{R}_{\dot{c}}(\dot{c}), J) = 0. \end{aligned}$$

This gives (14.15). Q.E.D.

Theorem 14.3.2 (Cartan-Hadamard [Aus]) *Let (M, F) be a positively complete Finsler space. Suppose that F has nonpositive curvature $\mathbf{K} \le 0$. Then $\exp_x$ is non-singular on T_xM for any $x \in M$.*

Proof: Let $y \in T_xM$ be an arbitrary unit vector and $c(t) = \exp_x(ty)$, $0 \le t < \infty$. For a vector $v \in T_xM$, let

$$J(t) := d(\exp_x)_{|ty}(tv).$$

J is the Jacobi field along c with $J(0) = 0$ and $\mathrm{D}_{\dot{c}}J(0) = v$. By the Gauss Lemma, we have

$$J(t) = J^{\perp}(t) + (at + b)\dot{c}(t),$$

where $J^{\perp}$ is a Jacobi field $g_{\dot{c}}$-orthogonal to $\dot{c}$. Note that $J = J^{\perp}$ if and only if v is g_y-orthogonal to y. In order to study the zeros of J, it suffices to study that of $J^{\perp}$. Thus we may assume that v is g_y-orthogonal to y. Let

$$f := \sqrt{g_{\dot{c}}(J, J)}.$$

Let $r_o > 0$ and assume that $J(r) > 0$ for $0 < r < r_o$. Observe that on $(0, r_o)$, f satisfies

$$f' = \frac{g_{\dot c}(\mathrm{D}_{\dot c}J, J)}{f}.$$

The Schwarz inequality gives

$$(f')^2 \le g_{\dot c}(\mathrm{D}_{\dot c}J, \mathrm{D}_{\dot c}J). \tag{14.16}$$

We also have

$$f(t) = t\sqrt{g_y(v,v)} + o(t),$$

and

$$\int_0^r (f')^2 dt \ge \frac{f(r)^2}{r}. \tag{14.17}$$

It follows from (14.16) and (14.17) that

$$\begin{aligned} f(r)f'(r) &= g_{\dot c(r)}(\mathrm{D}_{\dot c}J(r), J(r)) \\ &= \int_0^r \Big\{ g_{\dot c}(\mathrm{D}_{\dot c}J, \mathrm{D}_{\dot c}J) - g_{\dot c}(\mathbf{R}_{\dot c}(J), J) \Big\} dt \\ &\ge \int_0^r (f')^2 dt \ge \frac{f(r)^2}{r}. \end{aligned}$$

This implies

$$\frac{d}{dr}\Big(\ln \frac{f(r)}{r} \Big) \ge 0. \tag{14.18}$$

Hence the quotient $f(r)/r$ is non-decreasing on $(0, r_o)$. We obtain

$$\sqrt{g_{\dot c(r)}(J(r), J(r))} \ge r\sqrt{g_y(v,v)}, \qquad 0 < r < r_o. \tag{14.19}$$

From (14.19), we conclude that $J(r) = d(\exp_x)_{|ry}(rv) \ne 0$ for all $0 < r < \infty$. Hence $\exp_x$ is non-singular along ry, $0 < r < \infty$. Q.E.D.

The Cartan-Hadamard theorem indicates that for a positively complete Finsler space with nonpositive curvature, $\exp_x : T_xM \to M$ is a local diffeomorphism. With further arguments, we see that $\exp_x$ is actually a projection map or covering map. Thus if M is simply connected, then $\exp_x$ is a diffeomorphism. In this case, every pair of points $p, q \in M$ can be joined by an unique minimizing geodesic. See [BaChSh1] for more details.

Our next goal is to study positively complete Finsler spaces with uniformly positive (Ricci) curvature. First, we introduce the index form for vector fields along a geodesic. Let $c(t), 0 \le t \le r$ be a unit speed geodesic in a Finsler space (M, F). For a vector field $V(t)$ along c, define

$$\mathcal{I}(V,V) := \int_0^r \Big\{ g_{\dot c}(\mathrm{D}_{\dot c}V, \mathrm{D}_{\dot c}V) - g_{\dot c}\Big(\mathbf{R}_{\dot c}(V), V\Big) \Big\} dt.$$

Observe that if $E(t)$ is a parallel vector field along c with $g_{\dot c}(E,E)=1$ and $f(t)$ is a C^∞ function with $f(0)=0$, then

$$\mathcal{I}(fE,fE)=\int_0^r \Big\{(f')^2-g_{\dot c}\Big(\mathbf{R}_{\dot c}(E),E\Big)f^2\Big\}dt.$$

Lemma 14.3.3 *Let $y\in T_xM$ be a unit vector and $c(t):=\exp_x(ty)$. Suppose that $\exp_x$ is non-singular at $ty\in T_xM$ for all $0<t\le r$. Then for any Jacobi field $J(t)$ and any vector field $V(t)$ along c with*

$$V(0)=J(0)=0,\qquad V(r)=J(r),$$

the index forms satisfy

$$\mathcal{I}(V,V)\ge\mathcal{I}(J,J). \tag{14.20}$$

The equality holds if and only if $V=J$. Hence for any non-trivial vector field $V(t)$ along c with $V(0)=V(r)=0$,

$$\mathcal{I}(V,V)>0. \tag{14.21}$$

The proof is elementary, so it is omitted. See [BaChSh1].

Now we assume that $\exp_x$ is non-singular at ty for all $0<t\le r$. Take an arbitrary basis $\{e_i\}_{i=1}^n$ for T_xM with $e_1=y$ and $\{J_i(t)\}_{i=1}^n$ the Jacobi fields along c with $J_i(0)=0$ and $\mathrm{D}_{\dot c}J_i(0)=e_i$. Fix an arbitrary function $f(t)$ with $f(0)=0=f(r)$. By Lemma 14.3.3, for each $i=2,\cdots,n$,

$$\mathcal{I}(fE_i,fE_i)=\int_0^r \Big\{(f')^2-g_{\dot c}\Big(\mathbf{R}_{\dot c}(E_i),E_i\Big)f^2\Big\}dt>0. \tag{14.22}$$

Adding them together, one obtains

$$\int_0^r \Big\{(n-1)(f')^2-\mathbf{Ric}(\dot c)f^2\Big\}dt>0. \tag{14.23}$$

Assume that $\mathbf{Ric}(y)\ge(n-1)F^2(y)$. Then (14.23) is simplified to

$$\int_0^r \Big\{(f')^2-f^2\Big\}dt>0. \tag{14.24}$$

(14.24) holds for any C^∞ function $f\ne0$ with $f(0)=0=f(r)$. We claim that $0<r\le\pi$. Suppose this claim is false. Then for some $\varepsilon>0$, $\exp_x$ is non-singular at ty for all $0<t\le r:=(1+\varepsilon)\pi$. Let

$$f(t)=\sin\Big(\frac{t}{1+\varepsilon}\Big).$$

Plugging $f(t)$ into (14.24), one obtains

$$(1+\varepsilon)\Big[\frac{1}{(1+\varepsilon)^2}-1\Big]=\int_0^r\Big\{\frac{1}{(1+\varepsilon)^2}\cos^2\Big(\frac{t}{1+\varepsilon}\Big)-\sin^2\Big(\frac{t}{1+\varepsilon}\Big)\Big\}dt>0.$$

Clearly, this is impossible, because that the left hand side is negative. Thus the claim is true. We have proven the following

Theorem 14.3.4 (Bonnet-Myers [Aus]) *Let (M, F) be an n-dimensional positively complete Finsler space. Suppose that*

$$\mathbf{Ric}(y) \geq (n-1)F^2(y), \qquad \forall y \in TM \setminus \{0\}.$$

Then for any unit vector $y \in T_xM$, the conjugate value satisfies $\mathbf{c}_y \leq \pi$.

Let (M, F) be a Finsler space as in Theorem 14.3.4. For a point $x \in M$, define

$$\mathrm{Diam}(M, x) := \sup_{z \in M} d(x, z).$$

We can show that

$$\mathrm{Diam}(M, x) \leq \pi.$$

Consider the lifted Finsler metric on the universal covering $(\tilde{M}, \tilde{x}) \to (M, x)$. $\tilde{F}$ is also positively complete and satisfies the same Ricci curvature bound. Thus

$$\mathrm{Diam}(\tilde{M}, \tilde{x}) \leq \pi.$$

In particular, $\tilde{M}$ is compact. We conclude that the fundamental group of M is finite. See [BaChSh1] for more details.

14.4 Jacobi Fields in Isotropic Spray Spaces

One of the important problems in Spray Geometry is to understand the relationship between the Riemann curvature and the global geometrical/topological structures of a spray space. A natural approach is to study the behavior of the exponential map. When the spray is isotropic, the relationship between the Riemann curvature and the exponential map becomes much simpler.

Let $(M, \mathbf{G})$ be an isotropic spray space. Namely, the Riemann curvature $\mathbf{R}_y$ has the following form

$$\mathbf{R}_y(u) = R(y)u + \tau_y(u)\, y, \qquad u \in T_xM, \tag{14.25}$$

where $\tau_y \in T_x^*M$ satisfies

$$\tau_y(y) = -R(y). \tag{14.26}$$

Let $c(t)$ be a geodesic of $\mathbf{G}$ and $R(t) := R(\dot{c}(t))$. Let $\mathbf{s}(t)$ and $\mathbf{c}(t)$ denote the solutions of the following equation

$$y''(t) + R(t)y(t) = 0, \tag{14.27}$$

with

$$\mathbf{s}(0) = 0, \;\; \mathbf{s}'(0) = 1, \qquad \mathbf{c}(0) = 1, \;\; \mathbf{c}'(0) = 0.$$

Lemma 14.4.1 *For any Jacobi field J along c, there are two parallel vector fields $E(t)$ and $E^*(t)$ along c, which are linearly independent of $\dot{c}(t)$, such that*

$$J(t) = \mathbf{s}(t)E(t) + \mathbf{c}(t)E^*(t) + \varphi(t)\dot{c}(t), \tag{14.28}$$

where

$$\varphi(t) = -\iint \tau_{\dot{c}(t)}\Big(E(t)\Big)\mathbf{s}(t)dt - \iint \tau_{\dot{c}(t)}\Big(E^*(t)\Big)\mathbf{c}(t)dt.$$

Proof: Take an arbitrary parallel frame $\{E_i(t)\}_{i=1}^n$ along c with $E_1(t) = \dot{c}(t)$. Put

$$J(t) = J^i(t)E_i(t).$$

With (14.25) and (14.26), the Jacobi equation simplifies to

$$\frac{d^2J^1}{dt^2} + \tau_a(t)J^a(t) = 0, \tag{14.29}$$

$$\frac{d^2J^a}{dt^2} + R(t)J^a(t) = 0, \quad a = 2, \cdots, n, \tag{14.30}$$

where

$$\tau_a(t) := \tau_{\dot{c}(t)}\Big(E_a(t)\Big), \quad R(t) := R\Big(\dot{c}(t)\Big).$$

From (14.30), we obtain

$$J^a(t) = A^a\, \mathbf{s}(t) + B^a\, \mathbf{c}(t), \tag{14.31}$$

where A^a and B^a are constants. Let

$$E(t) := A^aE_a(t), \quad E^*(t) := B^aE_a(t).$$

Both $E(t)$ and $E^*(t)$ are parallel along c. Plugging (14.31) into (14.29) yields

$$\frac{d^2J^1}{dt^2} + \tau_{\dot{c}(t)}\Big(E(t)\Big)\, \mathbf{s}(t) + \tau_{\dot{c}(t)}\Big(E^*(t)\Big)\, \mathbf{c}(t) = 0.$$

This completes the proof. Q.E.D.

Assume that $\mathbf{G}$ is positively complete. Let c be a geodesic with $\dot{c}(0) = y \in T_xM$. For a vector $v \in T_xM$, let

$$J(t) := d(\exp_x)_{|ty}(tv), \qquad t \geq 0.$$

J satisfies

$$J(0) = 0, \qquad \mathrm{D}_yJ(0) = v. \tag{14.32}$$

Assume that v is linearly independent of y. Let $E(t)$ be the parallel vector field along c with $E(0) = v$. Take a parallel frame $\{E_i(t)\}_{i=1}^n$ along c with

$$E_1(t) := \dot{c}(t), \qquad E_2(t) = E(t).$$

Put

$$J(t) = J^i(t)E_i(t).$$

(14.32) implies

$$J^i(0) = 0, \quad \frac{dJ^i}{dt}(0) = \delta_{i2}.$$

From (14.29) and (14.30), we obtain

$$J(t) = \mathbf{s}(t)\ E(t) - \Big[\int_0^t \int_0^s \tau_{\dot c(\rho)}\Big(E(\rho)\Big)\mathbf{s}(\rho)\ d\rho ds\Big]\ \dot c(t), \tag{14.33}$$

where $\mathbf{s}(t)$ is defined in (14.27).

Suppose that

$$\mathbf{Ric} = (n-1)R \leq 0.$$

Then by an elementary argument

$$\mathbf{s}(t) \geq t, \quad \forall t > 0.$$

We conclude that

$$d(\exp_x)_{|ty}(tv) = J(t) \neq 0, \quad t > 0.$$

This proves the following

Theorem 14.4.2 *Let $(M, \mathbf{G})$ be a positively complete isotropic spray space. Suppose that*

$$\mathbf{Ric} \leq 0.$$

Then for any $x \in M$, $\exp_x : T_xM \to M$ is a local diffeomorphism.

Now we consider the case when $\mathbf{Ric} > 0$. In general, from (14.27), we do not know whether or not the solution $\mathbf{s}(t)$ has a zero in $(0, \infty)$. Suppose that $\mathbf{s}(t) = 0$ for some $t > 0$. Let r denote the first zero of $\mathbf{s}(t)$. It follows from (14.33) that

$$d(\exp_x)_{|ry}(v) = -\Big[\int_0^r \int_0^s \tau_{\dot c(\rho)}\Big(E(\rho)\Big)\mathbf{s}(\rho)\ d\rho ds\Big]\ \dot c(r), \tag{14.34}$$

where $E(t)$ is the parallel vector field along c with $E(0) = v$. Let

$$\Theta(v) := -\int_0^r \int_0^s \tau_{\dot c(\rho)}\Big(E(\rho)\Big)\mathbf{s}(\rho)\ d\rho ds.$$

Then $\Theta : T_xM \to \mathrm{R}$ is a linear form. It follows from (14.26) that

$$\tau_{\dot c}(\dot c) = -R(\dot c).$$

Thus

$$\Theta(y) = \int_0^r \int_0^s R(\dot{c}(\rho))\mathbf{s}(\rho)\, d\rho ds > 0.$$

Let

$$\mathrm{W}_y := \ker \Theta.$$

The above argument shows that W_y is a hyperplane and $y \notin \mathrm{W}_y$. From (14.34), we see that $d(\exp_x)_{|ry} = 0$ on W_y. This proves (i) and (ii) in the following

Theorem 14.4.3 *Let $(M, \mathbf{G})$ be positively complete isotropic spray space. Suppose that*

$$\mathbf{Ric} > 0.$$

Then for a non-zero vector $y \in T_xM$, one of the following holds

(i) $\exp_x$ *is non-singular along $x(t) = ty$, $0 \le t < \infty$,*

(ii) there is a number $r > 0$ and a hyperplane $\mathrm{W}_y \subset T_xM$ such that $d(\exp_x)_{|ry} = 0$ on W_y.

If along $c(t) = \exp_x(ty)$, $t \ge 0$,

$$\mathbf{Ric}(\dot{c}(t)) \ge (n-1)\lambda > 0. \tag{14.35}$$

Then only case (ii) occurs at $r \le \frac{\pi}{\sqrt{\lambda}}$.

Proof: Suppose the Ricci curvature satisfies the bound (14.35) along a geodesic c. Then the function $R(t) := R(\dot{c}(t))$ in (14.27) satisfies

$$R(t) \ge \lambda > 0.$$

Assume that $\mathbf{s}(t)$ is positive on an interval $(0, r)$. By an elementary argument, one can show that the solution $\mathbf{s}(t)$ satisfies

$$\mathbf{s}(t) \le \frac{\sin(\sqrt{\lambda} t)}{\sqrt{\lambda}}, \qquad \forall 0 < t \le r.$$

This implies $r \le \pi/\sqrt{\lambda}$. Q.E.D.

Recall that a spray $\mathbf{G}$ is said to be weakly Ricci-constant if the Ricci curvature $\mathbf{Ric} = (n-1)R$ satisfies

$$R_{;m}y^m := y^m R_{x^m} - 2G^m R_{y^m} = 0. \tag{14.36}$$

By Lemma 8.1.9, the Ricci curvature $\mathbf{Ric}(\dot{c}(t)) = \lambda$ is constant along any geodesic c. By the same argument as for Theorem 14.4.3, we immediately obtain the following

Corollary 14.4.4 *Let $(M, \mathbf{G})$ be a positively complete isotropic spray space. Suppose that $\mathbf{G}$ is weakly Ricci-constant. Then for any $y \in T_xM \setminus \{0\}$ with $R(y) = 1$, there is a hyperplane $\mathrm{W}_y \subset T_xM$ such that*

$$d(\exp_x)|_{|\pi y}\Big(\mathrm{W}_y\Big) = 0. \tag{14.37}$$

Example 14.4.1 Consider the following spray on R^2

$$\mathbf{G} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - v\sqrt{u^2+v^2}\,\frac{\partial}{\partial u} + u\sqrt{u^2+v^2}\,\frac{\partial}{\partial v}. \tag{14.38}$$

A direct computation using (8.42) or (8.44) yields

$$R^1_1 = v^2, \qquad R^1_2 = -uv, \qquad R^2_1 = -vu, \qquad R^2_2 = u^2.$$

Thus the Ricci curvature is given by

$$R(u,v) = R^1_1 + R^2_2 = u^2 + v^2 > 0. \tag{14.39}$$

Moreover, $\mathbf{G}$ is weakly Ricci-constant, i.e.,

$$R_{;m}y^m = R_x u + R_y v - v\sqrt{u^2+v^2}\;R_u + u\sqrt{u^2+v^2}\;R_v = 0.$$

By Corollary 14.4.4, we know that $\exp_0$ is singular on the tangent sphere $u^2 + v^2 = \pi^2$. We are going to verify this fact directly. By Example 14.1.1, the exponential map $\exp_0 : T_0\mathrm{R}^2 = \mathrm{R}^2 \to \mathrm{R}^2$ is given by

$$\exp_0(u,v) = \Big(\Phi(u,v), \Psi(u,v)\Big),$$

where

$$\begin{aligned}
\Phi(u,v) &= \frac{1}{\sqrt{u^2+v^2}}\Big(v\cos\sqrt{u^2+v^2} + u\sin\sqrt{u^2+v^2} - v\Big), & (14.40)\\
\Psi(u,v) &= \frac{1}{\sqrt{u^2+v^2}}\Big(-u\cos\sqrt{u^2+v^2} + v\sin\sqrt{u^2+v^2} + u\Big). & (14.41)
\end{aligned}$$

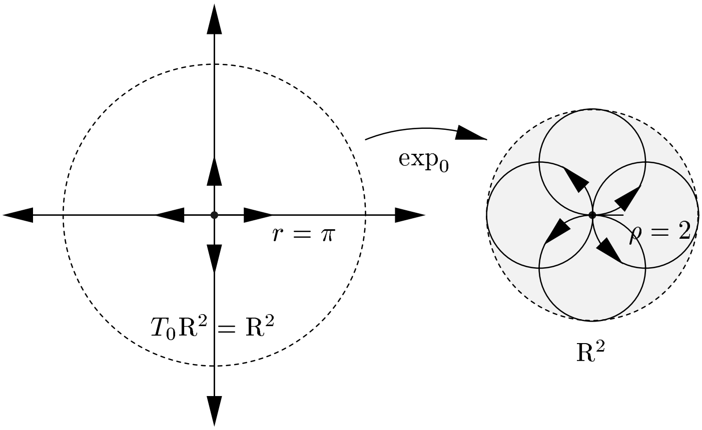
A direct computation yields

$$\Phi_u\Psi_v - \Phi_v\Psi_u = \frac{\sin\sqrt{u^2+v^2}}{\sqrt{u^2+v^2}}.$$

Thus $\exp_0$ is singular along circles of radius $k\pi$ in $T_0\mathrm{R}^2 = \mathrm{R}^2$, where $k = 1, 2, \cdots$. Observe that

$$\exp_0(u,v) = \frac{2}{\pi}(-v,u), \qquad u^2+v^2 = \pi^2.$$

$\exp_0$ maps the circle of radius $r = \pi$ in $T_0\mathrm{R}^2 = \mathrm{R}^2$ onto the circle of radius $\rho = 2$ in R^2.



♯

According to Lemma 8.1.10, two-dimensional sprays are always isotropic. Thus we can apply the above results to two-dimensional sprays associated with a SODE.

Let $\Phi = \Phi(x, y, \xi)$ be a C^∞ function on $\mathrm{R}^2 \times \mathrm{R}$. Consider the following equation on R

$$\frac{d^2y}{ds^2} = \Phi\Big(x, y, \frac{dy}{dx}\Big). \tag{14.42}$$

To study (14.42), we construct a homogeneous system on R^2

$$\begin{cases} \frac{d^2x}{dt^2} + 2G\Big(x, y, \frac{dx}{dt}, \frac{dy}{dt}\Big) = 0 \\ \frac{d^2y}{dt^2} + 2H\Big(x, y, \frac{dx}{dt}, \frac{dy}{dt}\Big) = 0 \end{cases} \tag{14.43}$$

where G and H satisfy

$$\Phi(x, y, \xi) = 2\xi G(x, y, 1, \xi) - 2H(x, y, 1, \xi). \tag{14.44}$$

By Lemma 3.1.1, we know that the graphs of solutions of (14.42) in R^2 coincide with the solutions of (14.43) in R^2 as point sets. Thus, we can study the solutions of (14.42) by investigating the geodesics of the corresponding spray on R^2

$$\mathbf{G} := u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - 2G(x, y, u, v)\frac{\partial}{\partial u} - 2H(x, y, u, v)\frac{\partial}{\partial v}, \tag{14.45}$$

By (8.42), the Ricci curvature $R = \mathbf{Ric}$ is given by

$$\begin{aligned} R \;=\; & 2G_x + 2H_y - G_u^2 - H_v^2 - 2H_uG_v - u(G_u + H_v)_x \\ & -v(G_u + H_v)_y + 2G(G_u + H_v)_u + 2H(G_u + H_v)_v. \end{aligned} \tag{14.46}$$

Suppose that $\mathbf{G}$ is a positively complete spray on R^2 with (3.13) such that the Ricci curvature $\mathbf{Ric} = R \le 0$. By Theorem 14.4.2, we know that for any $x \in \mathrm{R}^2$, the exponential map $\exp_{(x,y)} : T_{(x,y)}\mathrm{R}^2 \to \mathrm{R}^2$ is a local diffeomorphism.

Example 14.4.2 Consider a equation (14.42) where

$$\Phi(x,y,\xi) := A(x,y) + B(x,y)\xi + C(x,y)\xi^2 + D(x,y)\xi^3, \tag{14.47}$$

where A, B, C, D are C^∞ functions on R^2. There are several homogeneous systems (14.43) which we can use to study the equation (14.47) by Lemma 3.1.1. Below are some natural choices for (14.43).

Option 1: We take

$$\begin{cases} G & = \frac{1}{2}D(x,y)v^2 \\ H & -\frac{1}{2}\Big(A(x,y)u^2 + B(x,y)uv + C(x,y)v^2\Big) \end{cases} \tag{14.48}$$

By (14.46), we obtain

$$\begin{aligned} R &= \Big(AC + \frac{1}{2}B_x - \frac{1}{4}B^2 - A_y\Big)u^2 + (2AD - \frac{1}{2}B_y + C_x\Big)uv \\ &\quad + \Big(\frac{1}{2}BD + D_x\Big)v^2. \end{aligned} \tag{14.49}$$

Option 2: We take

$$\begin{cases} G: & = \frac{1}{2}\Big(B(x,y)u^2 + C(x,y)uv + D(x,y)v^2\Big) \\ H: & = -\frac{1}{2}A(x,y)u^2. \end{cases} \tag{14.50}$$

By (14.46), we obtain

$$\begin{aligned} R &= \Big(\frac{1}{2}AC - A_y\Big)u^2 + \Big(\frac{1}{2}C_x + 2AD - B_y\Big)uv \\ &\quad + \Big(D_x - \frac{1}{2}C_y + BD - \frac{1}{4}C^2\Big)v^2. \end{aligned} \tag{14.51}$$

Option 3: We take

$$\begin{cases} G & = \frac{1}{6}B(x,y)u^2 + \frac{1}{3}C(x,y)uv + \frac{1}{2}D(x,y)v^2 \\ H & = -\frac{1}{2}A(x,y)u^2 - \frac{1}{3}B(x,y)uv - \frac{1}{6}C(x,y)v^2 \end{cases} \tag{14.52}$$

By (14.46), we obtain

$$\begin{aligned} R &= \Big(\frac{1}{3}B_x - A_y - \frac{2}{9}B^2 + \frac{2}{3}AC\Big)u^2 + \Big(\frac{2}{3}C_x - \frac{2}{3}B_y \\ &\quad - \frac{4}{9}BC + 2AD\Big)uv + \Big(D_x - \frac{1}{3}C_y - \frac{2}{9}C^2 + \frac{2}{3}BD\Big)v^2. \end{aligned} \tag{14.53}$$

Option 4: Assume that $D = 0$. Besides the above options, we can also take

$$\begin{cases} G: & = \frac{1}{2}B(x,y)u^2 + C(x,y)uv \\ H: & = -\frac{1}{2}A(x,y)u^2 + \frac{1}{2}C(x,y)v^2. \end{cases} \tag{14.54}$$

By (14.46), we obtain

$$R = -A_y u^2 - B_y uv - C_y v^2. \tag{14.55}$$

For any option, if $R \leq 0$, then by Theorem 14.4.2, we know that for any $x \in \mathrm{R}^2$, the exponential map of the corresponding spray $\mathbf{G}$ is a local diffeomorphism.

$$\exp_{(x,y)} : T_{(x,y)}\mathrm{R}^2 \to \mathrm{R}^2.$$

♯

14.5 Finsler Spaces of Constant Curvature

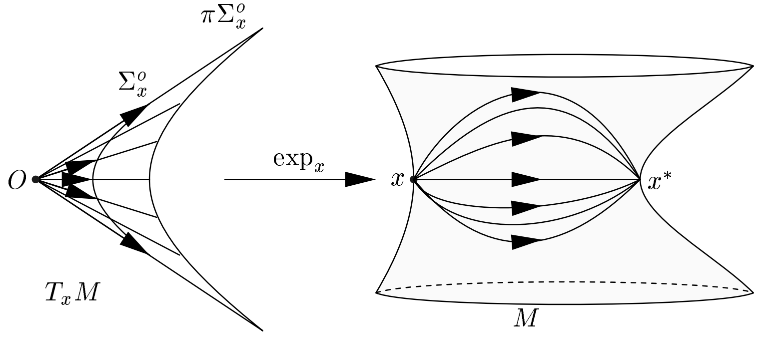
Although there are infinitely many positively complete Finsler spaces of constant curvature λ, the topology of the manifold M is quite special. By Theorem 14.4.2 we know that if $\lambda \leq 0$, the exponential map $\exp_x : T_xM \to M$ is a local diffeomorphism. By Proposition 14.4.3, we know that if $\lambda > 0$, the exponential map $\exp_x : T_xM \to M$ is singular in all directions, provided that the Finsler metric $L > 0$ on $TM\backslash\{0\}$. In this section, we will apply the techniques developed in the previous sections to study Finsler spaces of positive constant curvature.

Proposition 14.5.1 *Let (M, L) be a positively complete Finsler space of constant curvature $\lambda \neq 0$. For a point $x \in M$, let*

$$\Sigma_x := \Big\{ y \in T_xM,\ \lambda L(y) = 1 \Big\}. \tag{14.56}$$

Then for any connected component $\Sigma_x^o \subset \Sigma_x$,

$$\exp_x \Big(\pi \Sigma_x^o \Big) = x^*.$$



Proof: For a vector $y \in \Sigma_x$, let

$$y^\perp := \Big\{ w \in T_xM,\ g_y(y,w) = 0 \Big\}.$$

Since

$$L(y) = g_y(y,y) \neq 0,$$

$y^\perp$ is a hyperplane which does not contain y. For an arbitrary vector $w \in T_xM$ satisfying $d(\exp_x)|_{\pi y}(w) = 0$, put

$$w = w_1 + ry, \qquad w_1 \in y^\perp.$$

Let

$$J_1(t) := d(\exp_x)_{|ty}(tw_1)$$

It follows from the Gauss Lemma that

$$g_{\dot c(t)}\Big(J_1(t), \dot c(t)\Big) = g_y(w_1, y)t = 0.$$

Thus

$$\begin{aligned} 0 &= g_{\dot c(\pi)}\Big(d(\exp_x)_{|\pi y}(w),\ \dot c(\pi)\Big) \\ &= g_{\dot c(\pi)}\Big(J_1(\pi) + r\dot c(\pi),\ \dot c(\pi)\Big) \\ &= \lambda L(\dot c(\pi)). \end{aligned}$$

This implies $r = 0$. Thus $w \in y^\perp$. That is,

$$\ker \Big[d(\exp_x)|_{\pi y} \Big] = y^\perp.$$

Take an arbitrary curve $y(s)$ in Σ_x and let

$$\sigma(s) = \exp_x \Big(\pi y(s)\Big).$$

Then

$$y'(s) \in T_{y(s)}\Sigma_x = y(s)^\perp.$$

This implies

$$\dot\sigma(s) = d(\exp_x)_{|\pi y(s)}\Big(y'(s)\Big) = 0.$$

Thus

$$\sigma(s) = \text{a point}.$$

Q.E.D.

Let (M, L) be as in Proposition 14.5.1. Let

$$\Sigma_x^* := \Big\{ y \in T_xM,\quad \lambda L(y) = -1 \Big\}. \tag{14.57}$$

By a similar argument, one can prove that $\exp_x\Big(t\Sigma_x^*\Big)$ grows exponentially. More precisely, let $v \in T_y\Sigma_x^*$, then

$$d(\exp_x)_{|ty}(tv) = \sinh(t)\; E(t), \tag{14.58}$$

where $E(t)$ is a parallel vector field along $c(t) := \exp_x(ty)$ with $E(0) = v$.

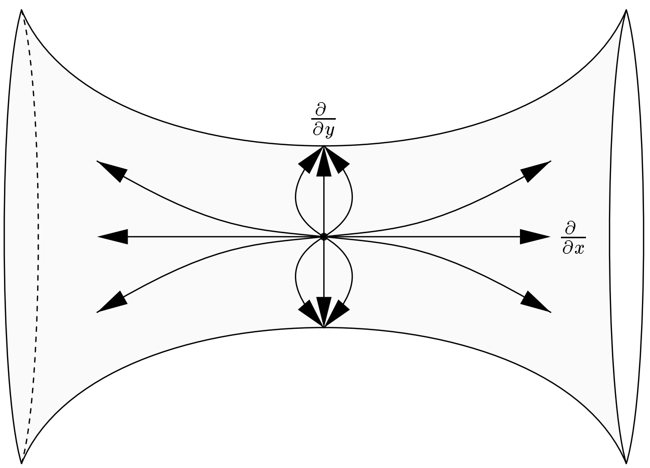
Example 14.5.1 Consider a Finsler metric defined on $M^2 = (-\infty, \infty) \times \mathrm{S}^1$,

$$L(x, y, u, v) = -\frac{1}{1+x^2}\; u^2 + \Big(1 + x^2\Big)\; v^2. \tag{14.59}$$

The Gauss curvature

$$\mathbf{K} = 1.$$

Since L is a pseudo-Finsler metric, (M, L) looks like a surface in the Euclidean space $\mathbb{R}^3$ with negative curvature.



♯

If a Finsler metric is positive definite, then the positivity of the Riemann curvature makes the space compact.

Proposition 14.5.2 *Let (M, F) be a positively complete positive definite Finsler space of constant curvature $\mathbf{R} = 1$. Then for any $x \in M$, there is a point $x^* \in M$ such that*

$$\exp_x\Big(\pi\; \mathrm{S}_x M\Big) = x^*, \tag{14.60}$$

where $\mathrm{S}_x M$ denotes the unit sphere of F_x in $T_x M$.

Proof: Since F is positive definite, the unit sphere

$$\mathrm{S}_x M = \Big\{ y \in T_x M,\ F_x(y) = 1 \Big\}$$

is a strongly convex closed hypersurface around the origin. Thus the set Σ_x in (14.56) is the whole indicatrix

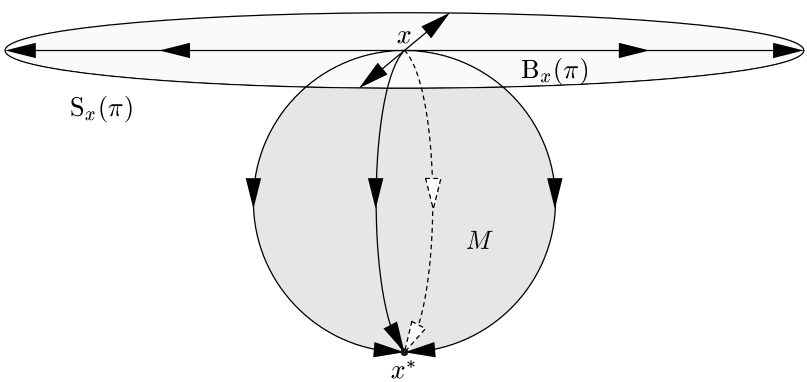
$$\Sigma_x = \mathrm{S}_x M.$$

It follows from Proposition 14.5.1 that there is a point $x^* \in M$ such that

$$\exp_x \Big(\pi \mathrm{S}_x M \Big) = x^*.$$

This implies that M must be compact with $d(x, x^*) \le \pi$.

Q.E.D.



Let F be a positive definite Finsler metric on S^n with constant curvature $\mathbf{R} = 1$. Fix a point $x \in \mathrm{S}^n$. There is an unique point $x^* \in \mathrm{S}^n$ with $d(x, x^*) = \pi$. The exponential map $\exp_x : T_x \mathrm{S}^n \to \mathrm{S}^n$ gives rise to a diffeomorphism

$$\varphi : (0, \pi) \times \mathrm{S}_x M \to \mathrm{S}^n - \{x, x^*\}$$

by

$$\varphi(t, y) := \exp_x(ty).$$

The induced Riemannian metric $\dot{g}_x$ on $\mathrm{S}_x M$ is given by

$$\dot{g}_x(u, v) := g_y(u, v), \qquad u, v \in T_y \mathrm{S}_x M \subset T_y M.$$

Let

$$Y := \varphi_* \Big(\frac{\partial}{\partial t} \Big).$$

Y is a unit geodesic field on S^n from x to x^*. Define

$$\hat{g} := g_Y.$$

$\hat{g}$ is a smooth Riemannian metric on $\mathrm{S}^n - \{x, x^*\}$. We claim that

$$\varphi^* \hat{g} = dt^2 \oplus \sin^2(t) \dot{g}_x. \tag{14.61}$$

Proof: Fix $y \in \mathrm{S}_x M$ and let $c(t) := \varphi(t, y)$. For any $u \in T_y \mathrm{S}_x M$, let

$$J_u(t) := d\varphi_{|(t,y)}(tu) = d(\exp_x)_{|ty}(tu).$$

J is a Jacobi field along c. By the Gauss Lemma, $J_u(t)$ is orthogonal to $\dot{c}(t)$ with respect to $\hat{g}$. Moreover

$$J_u(t) = \sin(t)\ E_u(t),$$

where $E_u(t)$ is parallel along c with $E_u(0) = u$ (here we view $T_y \mathrm{S}_x M$ as a subspace in $T_x M$. For a vector $v \in T_y \mathrm{S}_x M$, define J_v as above.

$$\begin{aligned}
\varphi^* \hat{g}(tu, tv) &= g_{\dot{c}(t)}(J_u(t), J_v(t)) \\
&= \sin^2(t) g_{\dot{c}(t)}(E_u(t), E_v(t)) \\
&= \sin^2(t) g_y(u, v) \\
&= \sin^2(t) \dot{g}_x(u, v).
\end{aligned}$$

This proves (14.61). Q.E.D.

We have to come to an end. But this is also a new beginning ...

Bibliography

[AbPa] M. Abate and G. Patrizio, *Finsler metrics-a global approach, with applications to geometric function theory*, Lecture Notes in Mathematics **1591**, Springer-Verlag, Berlin, 1994.

[Ai] T. Aikou, *Some remarks on the geometry of tangent bundles of Finsler spaces*, Tensor, N. S. **52**(1993), 234-242.

[Ale] R. Alexander, *Planes for which the lines are the shortest paths between points*, Illinois J. of Math., **22**(1978), 177-190.

[Alv1] J.C. Álvarez, *A symplectic construction of projective Finsler metric s on* R^n, preprint (1995).

[Alv2] J.C. Álvarez, *Contact topology, taut immersions, and Hilbert's fourth problem*, preprint.

[AlGeSm] J.C. Álvarez, I.M. Gelfand and M. Smirnov, *Crofton Densities, symplectic geometry and Hilbert's fourth problem*, Arnold-Gelfand Mathematical Seminars, Geometry and Singularity Theory, V.I. Arnold, I.M. Gelfand, M. Smirnov, and V.S. Retakh (eds.). Birkhauser, Boston, 1997, pp. 77-92.

[An1] P.L. Antonelli, *Finsler Volterra-Hamilton systems in ecology*, Tensor, N.S. **50**(1991), 22-31.

[An2] P.L. Antonelli, *On y-Berwald connections and Hutchinson's ecology of social-interactions,* Tensor, N. S. **52**(1993), 27-36.

[AnBu] P.L. Antonelli and I. Bucataru, *Volterra-Hamilton production models with discounting general theory and working examples*, to appear in Nonlinear analysis, Series A.

[AnBr] P.L. Antonelli and R. Bradbury, *Volterra-Hamilton Models in Ecology and Evolution of Colonial Organisms*, World Scientific Press, 1994.

[AnHaMo] P.L. Antonelli, B. Han and J. Modayil, *New results on 2-dimensional constant sprays with an application to heterochrony*, Nonlinear analysis B, **37**(1998), 545-566.

[AnInMa] P.L. Antonelli, R. Ingarden and M. Matsumoto, *The theory of sprays and Finsler spaces with applications in physics and biology*, Kluwer Academic Publishers, 1993.

[AnMa1] P.L. Antonelli and M. Matsumoto, *On conformal and projective flatness of two-dimensional constant-Berwald Finsler spaces in Epidemiology*, Open Systems & Information Dynamics, **3**(1995), 305-317.

[AnMa2] P.L. Antonelli and M. Matsumoto, *Two-dimensional Finsler spaces of locally constant connection*, Tensor, N. S. **57**(1996), 8-19.

[AnMaZa] P.L. Antonelli, M. Matsumoto and T. J. Zastawniak, *On y-Berwald spaces of dimension two and associated heterochronic systems with Finslerian noise*, Contemporary Mathematics **196**(1996), 203-212.

[AnZa1] P.L. Antonelli and T. Zastawniak (eds.), *Finsler geometry , Finsler Spaces and Noise Applied in Biology and Physics*, Pergamon Press, Mathematical and Computer Modeling, **20**(1994) no. 4/5.

[AnZa2] P.L. Antonelli and T. Zastawniak, *Fundamentals of Finslerian Diffusion with Applications*, Kluwer Academic Press, 1998.

[Ar] V.I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Grundlehren der mathematischen Wissenschaften 250, Springer-Verlag, Berlin Heidelberg New York, 1988.

[As1] G.S. Asanov, *Finsler geometry, relativity and gauge theories*, D. Reidel Publ. Comp., Dordrecht, 1985.

[As2] G.S. Asanov, *Finsler cases of GF-spaces* Aeguationes Mathematicae **49**(1995), 234-251.

[As3] G.S. Asanov, *Finslerian metric functions over the product $\mathbb{R} \times M$ and their potential applications*, Rep. on Math. Phys. **41**(1998), 117-131.

[Aus] L. Auslander, *On curvature in Finsler geometry*, Trans. Amer. Math. Soc. **79**(1955), 378-388.

[AZ1] H. Akbar-Zadeh, *Sur les sous-varietes des varietes Finsleriennes*, C. R. Acad. Sci. Paris Ser. A-B **266**(1968), A146-A148.

[AZ2] H. Akbar-Zadeh, *Sur les espaces de Finsler á courbures sectionnelles constantes*, Bull. Acad. Roy. Bel. Cl, Sci, 5e Série - Tome LXXXIV (1988) 281-322.

[BaCh] D. Bao and S. S. Chern, *On a notable connection in Finsler geometry*, Houston J. Math. **19**(1)(1993), 135-180.

[BaChSh1] D. Bao, S. S. Chern and Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Springer-Verlag, to appear.

[BaChSh2] D. Bao, S. S. Chern and Z. Shen, *Rigidity issues on Finsler surfaces*, Rev. Roumaine Math. Pures Appl. **42**(1997), 707-735.

[BaIlKi] S. Bácsó, F. Ilosvay and B. Kis, *Landsberg spaces with common geodesics*, Publ. Math. Debrecen **42**(1993), 139-144.

[BaMa1] S. Bácsó and M. Matsumoto, *Reduction theorems of certain Landsberg spaces to Berwald spaces*, Publ. Math. Debrecen, **48**(1996), 357-366.

[BaMa2] S. Bácsó and M. Matsumoto, *On Finsler spaces of Douglas type. A generalization of the notion of Berwald space*, Publ. Math. Debrecen, **51**(1997), 385-406.

[BaMa3] S. Bácsó and M. Matsumoto, *On Finsler spaces of Douglas type II. Projectively flat spaces*, Publ. Math. Debrecen, to appear.

[BaMa4] S. Bácsó and M. Matsumoto, *On Finsler spaces of Douglas type III. Remarks on projectively flat spaces*, preprint, 1998.

[BaMa5] S. Bácsó and M. Matsumoto, *Finsler spaces with the h-curvature tensor dependent on position alone,* Publ. Math. Debrecen, **55**(1999), 199-210.

[BaMa6] S. Bácsó and M. Matsumoto, *Randers spaces with the h-curvature tensor H dependent on position alone,* Publ. Math. Debrecen, to appear.

[BaSh1] D. Bao and Z. Shen, *On the volume of unit tangent spheres in a Finsler space*, Results in Math. **26**(1994), 1-17.

[BaSh2] D. Bao and Z. Shen, *Finsler metrics of constant curvature on the Lie group S^3*, preprint.

[Bj] A. Bejancu, *Finsler geometry and applications*, Ellis Horwood Limited, Chichester, 1990.

[Bk1] F. Brickell, *A theorem on homogeneous functions*, J. London Math. Soc. **42**(1967), 325-329.

[Bk2] F. Brickell, *A new proof of Deicke's theorem on homogeneous functions*, Proc. of AMS **16**(1965), 190-191.

[Br1] R. Bryant, *Finsler structures on the 2-sphere satisfying $K = 1$*, Finsler Geometry, Contemporary Mathematics **196**, Amer. Math. Soc., Providence, RI, 1996, 27-42.

[Br2] R. Bryant, *Projectively flat Finsler 2-spheres of constant curvature*, Selecta Math., New Series, **3**(1997), 161-204.

[Br3] R. Bryant, in preparation.

[Bu1] H. Busemann, *Intrinsic area*, Ann. of Math. **48**(1947), 234-267.

[Bu2] H. Busemann, *The foundations of Minkowskian geometry*, Comment. Math. Helv. **24**(1950), 156-187.

[Bu3] H. Busemann, *The geometry of geodesics*, Academic Press, New York, 1955.

[BuKe] H. Busemann and J. P. Kelly, *Projective geometry and projective Finsler metrics*, Academic Press, New York, 1953.

[BuMa] H. Busemann and W. Mayer, *On the foundations of the calculus of variations*, Trans. AMS. **49**(1941), 173-198.

[Bw1] L. Berwald, *Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus*, Math. Z. **25**(1926), 40-73.

[Bw2] L. Berwald, *Parallelúbertragung in allgemeinen Räumen*, Atti Congr. Intern. Mat. Bologna **4**(1928), 263-270.

[Bw3] L. Berwald, *Über die n-dimensionalen geometrien konstanter krümmung, in denen die geraden die kürzesten sind*, Math. Z. **30**(1930), 449-469.

[Bw4] L. Berwald, *Über eine charakteristische Eigenschaft der allgemeinen Räume konstanter Krümmung mit geradlinigen Extremalen*, Monatshefte für Mathematik und Physik **36**(1929), 315-330.

[Bw5] L. Berwald, *On Finsler and Cartan geometries III, Two-dimensional Finsler spaces with vectilinear extremals*, Ann. of Math. **42**(1941), 84-112.

[Bw6] L. Berwald, *Ueber die Beziehungen zwischen den Theorien der Parallelübertragung in Finslerschen Räumen*, Nederl. Akad. Wetensch. Proc. **49**(1946), 642-647.

[Bw7] L. Berwald, *Ueber Finslersche und Cartansche Geometrie IV. Projektivkrümmung allgemeiner affiner Räume und Finslersche Räume skalarer Krümmung*, Ann. of Math. **48**(1947), 755-781.

[Bw8] L. Berwald, *Über systeme von gewöhnlichen differentialgleichungen zweiter ordnung, deren integralkurven mit dem system der geraden linien topologisch äquivalent sind*, Ann. of Math. **48**(1947), 193-215.

[Ca] E. Cartan, *Les espaces de Finsler*, Actualités 79, Paris, 1934.

[Ch1] S.S. Chern, *On the Euclidean connections in a Finsler space*, Proc. National Acad. Soc., **29**(1943), 33-37; or Selected Papers, vol. II, 107-111, Springer 1989.

[Ch2] S.S. Chern, *Sur la Géométrie d'un systéme d'Équations Différentielles du second ordre*, Bull. Sci. Math. **63**(1939), 206-212.

[Ch3] S. S. Chern, *On Finsler geometry*, C. R. Acad. Sc. Paris **314**(1992), 757-761.

[CPST] M. Crampin, G.E. Prince, W. Sarlet and G. Thompson, *The inverse problem of the calculus of variations: separable systems*, preprint (1998).

[CSMBP] M. Crampin, W. Sarlet, E. Martínez, G.B. Byrnes and G.E. Prince, *Towards a geometrical understanding of Douglas's solution of the inverse problem of the calculus of variations*, Inverse Problems, **10**(1994), 245-260.

[Dar] G. Darboux, *Leçons sur la Theorie des Surfaces*, **3** (604/605), Gauthier-Villars, Paris, 1894.

[Dav1] D.R. Davis, *The inverse problem of the calculus of variations in higher space*, Trans. of Amer. Math. Soc. **30**(1928), 710-736.

[Dav2] D.R. Davis, *The inverse problem of the calculus of variations in a space of* $(n+1)$ *dimensions*, Bull. of Amer. Math. Soc. **35**(1929), 371-380.

[Dav3] D.R. Davis, *Integrals whose extremals are a given 2-parameter family of curves*, Trans. of Amer. Math. Soc. **33**(1931), 311-314.

[Daz1] P. Dazord, *Propriétés globales des géodesiques des espaces de Finsler*, Thèse, Universite de Lyon, 1969.

[Daz2] P. Dazord, *Tores Finslériens sans points conjugués*, Bull. Soc. Math. France, **99**(1971), 171-192.

[De] A. Deicke, *Über die Finsler-Räume mit* $A_i = 0$, Arch. Math. **4**(1953), 45-51.

[Dg1] J. Douglas, *The general geometry of paths*, Ann. of Math. **29**(1927-28), 143-168.

[Dg2] J. Douglas, *Solution of the inverse problem of the calculus of variations*, Trans. of Amer. Math. Soc. **50**(1941), 71-128.

[Dr1] V.S. Dryuma, *On geometry of the second order differential equations*, Proceedings of Conference: Nonlinear Phenomena, Moskva, Nauka, 1991, pages 41-47.

[Dr2] V.S. Dryuma, *Proceedings of the First Workshop of Nonlinear Physics: theory and experiment*, Word Scientific, Singapore, 1996, pages 83-93.

[Dr3] V.S. Dryuma, *On applications of Riemannian geometry in theory of the second order ordinary differential equations*, Buletinul A.S. a R.M. Mathematica, **31**(1999), 95-102.

[Du] C. Duran, *A volume comparison theorem for Finsler spaces*, Proc. of AMS **126**(1998), 3079-3082.

[Fi] P. Finsler, *Über Kurven und Flächen in allgemeinen Räumen,* (Dissertation, Göttingan, 1918), Birkhäuser Verlag, Basel, 1951.

[Fu1] P. Funk, *Über Geometrien bei denen die Geraden die Kürzesten sind*, Math. Ann. **101**(1929), 226-237.

[Fu2] P. Funk, *Über zweidimensionale Finslersche Räume, insbesondere über solche mit geradlinigen Extremalen und positiver konstanter Krümmung*, Math. Z. **40**(1936), 86-93.

[GrMu] J. Grifone and Z. Muzsnay, *Sur le problème inverse du calcul des variations: existence de lagrangiens associés à un spray dans le cas isotrope*, Ann. Inst. Fourier, **49**(4)(1999), 1384-1421.

[HaHoMa] M. Hashiguchi, S. Hojo and M. Matsumoto, *On Landsberg spaces of two dimensions with* (α, β)*-metric*, J. Korean Math. Soc. **10**(1973), 17-26.

[HaIc1] M. Hashiguchi and Y. Ichijyō, *On some special* (α, β) *metrics*, Rep. Fac. Sci. Kagoshima Univ. **8**(1975), 39-46.

[HaIc2] M. Hashiguchi and Y. Ichijyō, *On conformal transformations of Wagner spaces,* Rep. Fac. Sci. Kagoshima Univ. **10**(1977), 19-25.

[HaIc3] M. Hashiguchi and Y. Ichijyō, *Randers spaces with rectilinear geodesics*, Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. & Chen.), **13**(1980), 33-40.

[Ham] G. Hamel, *Über die Geometrieen in denen die Geraden die Kürzesten sind*, Math. Ann. **57**(1903), 231-264.

[Ic1] Y. Ichijyō, *Finsler spaces modeled on a Minkowski space*, J. Math. Kyoto Univ. **16**(1976), 639–652.

[Ic2] Y. Ichihyō, *On special Finsler connections with vanishing hv-curvature tensor*, Tensor, N. S. **32**(1978), 146-155.

[Ik] F. Ikeda, *On two-dimensional Landsberg spaces*, Tensor, N. S. **33**(1979), 43-48.

[In] R. Ingarden, *On physical applications of Finsler geometry*, Contemporary Mathematics **196**(1996), 213-222.

[Iz] H. Izumi, *Some problems in special Finsler spaces*, Symp. on Finsler Geom. at Kinosaki, Nov. 14-16, 1984.

[JiSh] M. Ji and Z. Shen, *On strongly convex indicatrices in Minkowski geometry*, Canadian Math. Bull., to appear.

[KawA] A. Kawaguchi, *On the theory of non-linear connections II, Theory of Minkowski spaces and of non-linear connections in a Finsler space*, Tensor, N. S. **6**(1956), 165-199.

[KawH] H. Kawaguchi, *On Finsler spaces with vanishing second curvature tensor*, Tensor, N. S. **26**(1972), 250-254.

[Ki] S. Kikuchi, *On the condition that a space with (α, β)-metric be locally Minkowskian*, Tensor, N. S. **33**(1979), 242-246.

[Kl] F. Klein, *Vorlesungen über Nicht-Euklidische Geometrie*, Springer, Berlin, 1928.

[Kn] M.S. Knebelman, *Collinations and motions in generalized spaces*, Amer. J. of Math. **51**(1927), 527-564.

[Ko1] D. Kosambi, *The existence of a metric and the inverse variational problem*, Bull. U. P. Acad. of Science, **2**.

[Ko2] D. Kosambi, *Parallelism and path-spaces*, Math. Z. **37**(1933), 608-618.

[Ko3] D. Kosambi, *Systems of differential equations of second order*, Quart. J. Math. (Oxford Ser.) **6**(1935), 1-12.

[La1] G. Landsberg, *Über die Totalkrümmung*, Jahresberichte der deut Math. Ver. **16**(1907), 36-46.

[La2] G. Landsberg, *Über die Krümmung in der variationsrechung*, Math. Ann. **65**(1908), 313-349.

[Lan] S. Lang, *Differential and Riemannian Manifolds*, Springer-Verlag New York Inc., 1995.

[Ma1] M. Matsumoto, *On C-reducible Finsler spaces*, Tensor, N. S. **24**(1972), 29-37.

[Ma2] M. Matsumoto, *On Finsler spaces with Randers metric and special forms of important tensors*, J. Math. Kyoto Univ. **14**(1974), 477-498.

[Ma3] M. Matsumoto, *Projective changes of Finsler metrics and projectively flat Finsler spaces*, Tensor, N. S. **34**(1980), 303-315.

[Ma4] M. Matsumoto, *Foundations of Finsler Geometry and special Finsler Spaces*, Kaiseisha Press, Japan 1986.

[Ma5] M. Matsumoto, *Theory of Y-extremal and minimal hypersurfaces in a Finsler space*, J. Math. Kyoto Univ. **26**(1986), 647-665.

[Ma6] M. Matsumoto, *Randers spaces of constant curvature*, Rep. on Math. Phys. **28**(1989), 249-261.

[Ma7] M. Matsumoto, *Contributions of prescribed supporting element and the Cartan Y-connection*, Tensor, N. S. **49**(1990), 9-17.

[Ma8] M. Matsumoto, *Finsler spaces of constant curvature with Kropina metric*, Tensor, N. S. **50**(1991), 194-201.

[Ma9] M. Matsumoto, *The Tavakol-van der Bergh conditions in the theories of gravity and projective changes of Finsler metrics*, Publ. Math. Debrecen **42**(1993), 155-168.

[Ma10] M. Matsumoto, *Geodesics of two-dimensional Finsler spaces*, Mathl. Comput. Modelling **20**(1994), 1-23.

[Ma11] M. Matsumoto, *Every path space of dimension two is projectively related to a Finsler space*, Open Syst. & Inform. Dynamics **3**(1995), 291-303.

[Ma12] M. Matsumoto, *Theory of Finsler spaces with m-th root of metric II*, Publ. Math. Debrecen, **49**(1996), 135-155.

[MaOk] M. Matsumoto and K. Okubo, *Theory of Finsler spaces with m-th root metric*, Tensor, N. S. **56**(1995), 93-104.

[MaSh1] M. Matsumoto and H. Shimada, *On Finsler spaces with* 1*-form metric*, Tensor, N. S. **32**(1978), 161-169.

[MaSh2] M. Matsumoto and H. Shimada, *On Finsler spaces with* 1*-form metric II*, Tensor, N. S. **32**(1978), 275-278.

[MaWe] M. Matsumoto and X. Wei, *Projective changes of Finsler spaces of constant curvature*, Publ. Math. Debrecen **44**(1994), 175-181.

[MePa] M. Meyer and A. Pajer, *On Santaló's inequality*, Geometric aspects of functional analysis, Springer Lecture Notes in Mathematics 1376, pp. 261-263.

[Mi1] R. Miron, *The geometry of higher-order Finsler spaces*, Hadronic Press, Inn., USA, 1998.

[Mik1] J. Mikes, *On geodesic mappings of Einsteinian spaces*, Mat. Zametki **28**(1980), no. 6, 935-939.

[Mik2] J. Mikes, Geodesic mappings of special Riemannian spaces, Colloq. Math. Soc. J. Bolyai, 46. Top. in Dif. Geom., DEbrecen, 1984, vol. 2, Amsterdam, 1988, pp. 793-813.

[MiAn] R. Miron and M. Anastasiei, *Vector bundles and Finsler spaces with applications to relativity*, Geometry Balkan Press, Bucharest, Romania, 1997.

[Mo] X. Mo, *Characterization and structure of Finsler spaces with constant flag curvature*, Science in China **41**(1998), 908-917.

[Nu] S. Numata, *On Landsberg spaces of scalar curvature*, J. Korea Math. Soc. **12**(1975), 97-100.

[Ok] T. Okada, *On models of projectively flat Finsler spaces of constant negative curvature*, Tensor, N. S. **40**(1983), 117-123.

[Po] V. Pogorelov, *Hilbert's Fourth Problem*, Translated by R. Silverman. Scripta Series in Mathematics, Winston and Sons, 1979.

[Rap] A. Rapcsák, *Über die bahntreuen Abbildungen metrisher Räume*, Publ. Math. Debrecen, **8**(1961), 285-290.

[Ru] H. Rund, *The Differential Geometry of Finsler Spaces*, Springer, 1959.

[SaSh] V. S. Sabau and H. Shimada, *On the classification of* (α, β)*-metric spaces*, preprint, 1998.

[Sc] R. Schneider, *Über die Finsleräume mit* $S_{ijkl} = 0$, Arch. Math. **19**(1968), 656-658.

[SCM] W. Sarlet, M. Crampin and E. Martínez, *The integrability Conditions in the Inverse Problem of the Calculus of Variations for Second-Order Ordinary Differential Equations*, Acta Appl. Math., to appear.

[Sh1] Z. Shen, *On a connection in Finsler geometry*, Houston J. of Math. **20**(1994), 591-602.

[Sh2] Z. Shen, *Finsler spaces of constant positive curvature*, Contemporary Math. **196**(1996), 83-92.

[Sh3] Z. Shen, *Volume comparison and its applications in Riemann-Finsler geometry*, Advances in Math. **128**(1997), 306-328.

[Sh4] Z. Shen, *On Finsler geometry of submanifolds*, Math. Ann. **311**(1998), 549-576.

[Sh5] Z. Shen, *Curvature, distance and volume in Finsler Geometry*, IHES preprint (1997).

[Sh6] Z. Shen, *Conjugate radius and positive scalar curvature*, Math. Z. (to appear)

[Sh7] Z. Shen, *On projectively related Einstein metrics in Riemann-Finsler geometry*, Math. Ann. (to appear).

[Sh8] Z. Shen, *On R-quadratic Finsler spaces*, Publicationes Math. Debrecen, (to appear).

[Sh9] Z. Shen, *Funk metrics and R-flat sprays*, preprint (2000).

[ShKi] C. Shibata and M. Kitayama, *On Finsler spaces of constant positive curvature*, Proc. of the Romanian-Japanese Colloquium on Finsler Geometry, Braşov, (1984), 139-157.

[SM] E. Socié-Méthou, *Metric properties of Hilbert geometry*, preprint.

[SSAY] C. Shibata, H. Shimada, M. Azuma and H. Yosuda, *On Finsler spaces with Randers' metric*, Tensor, N. S. 31(1977), 219-226.

[Sz1] Z. Szabó, *Positive definite Berwald spaces (Structure theorems on Berwald spaces)*, Tensor, N. S. **35**(1981), 25-39.

[Sz2] Z. Szabó, *Ein Finslerscher Raum ist gerade dann von skalarer Krümmung, wenn seine Weyl sche Projektivkrümmüng verschwindet*, Acta Sci. Math. (Szeged) **39**(1977), 163-168.

[Sz3] Z. Szabó, *Hilbert's Fourth Problem. I.* Advances in Mathematics, **59**(1986).

[Th1] T.Y. Thomas, *On projective and equi-projective geometries of paths*, Proc. N. A. S. **11**(1925), 199-203.

[Th2] T.Y. Thomas, *A projective theory of affinely connected manifolds*, Math. Z. **25**(1926), 723-733.

[Th3] T.Y. Thomas, *Lectures on Differential Geometry: Basic Topics in the Theory of Generalized Spaces*, Planographed by Edwards Brothers, Inc., Ann Arbor, Michigan, 1938.

[Tho] A.C. Thompson, *Minkowski Geometry*, Encyclopedia of Math. and Its Applications, Vol. 63, Cambridge Univ. Press, Cambridge, 1996.

[Tr] A. Tresse, *Sur les invariants differentiels des groupes continues des transformations*, Acta Math. **18**(1894), 1-88.

[Va] O. Varga, *Die Krümmung der Eichfläche des Minkowskischen Raumes und die geometrische Deutung des einen Krümmungstensors des Finslerschen Raumes*, jour Abh. Mth. Sem. Univ. Hamburg **20**(1955), 41-51.

[Ve1] O. Veblen, *Remarks on the foundations of geometry*, Bull. of AMS **31**(1925), 121-141.

[Ve2] O. Veblen, *Generalized projective geometry*, J. London Math. Soc. **4**(1929), 140-160.

[VeTh1] O. Veblen and J. M. Thomas, *The geometry of paths*, Trans. Amer. Math. Soc. **25**(1923), 551-608.

[VeTh2] O. Veblen and J. M. Thomas, *Projective invariants of affine geometry of paths*, Ann. of Math. **27**(1926), 293-.

[Wa1] V.V. Wagner, *On generalized Berwald spaces*, C. R. Dokl. Acad. Sci. URSS, N. S. **39**(1943), 3-5.

[Wa2] V.V. Wagner, *Two-dimensional spaces with the metric defined by a cubic differential form*, (Russian and English), Saratow I **35**(1938), no. 14, 29-40.

[We1] H. Weyl, *Zur Infinitesimalgeometrie*, Göttinger Nachrichten, (1921), 99-112.

[We2] H. Weyl, *Gravitation und Elektrizität*, Sitzungsberichte Akademie der Wissenschaften Berlin, 465-480.

[Wh] J.H. C. Whitehead, *Convex regions in the geometry of paths*, Quart. J. Math. Oxford Ser. 3 (1932), 33-42.

[YaSh] H. Yasuda and H. Shimada, *On Randers spaces of scalar curvature*, Rep. on Math. Phys. **11**(1977), 347-360.

[Za] E.M. Zaustinsky, *Spaces with non-symmetric distance*, Mem. Amer. Math. Soc. **34**(1959).

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